Introduction to Part II

The utility of crossed modules for certain nonabelian homotopical calculations in dimension 2 has been shown in Part I, mainly as applications of a 2-dimensional Seifert–van Kampen Theorem. In Part II, we obtain homotopical calculations using *crossed complexes*, which are a kind of combination of crossed modules of groupoids with chain complexes, but keeping the operations in all dimensions. Again, a Higher Homotopy Seifert–van Kampen Theorem (HHSvKT) plays a key role, but we have to cover also a range of new techniques.

Chapter 7 sets out the basic structures we need to consider, including their important relation with chain complexes with a groupoid of operators. Also included is a brief account of homotopies, an area which is developed fully in Chapter 9.

Chapter 8 is devoted to the statement and immediate applications of the Higher Homotopy Seifert–van Kampen Theorem for crossed complexes.

Chapter 9 introduces a crucial monoidal closed structure on the category of crossed complexes. This structure gives notions of homotopy and higher homotopy for crossed complexes.

Chapter 10 develops the notion of *free crossed resolution* of a group or groupoid, including an outline of a method of computation for finitely presented finite groups, by the method of 'constructing a home for a contracting homotopy'. This uses the notion of covering morphism of crossed complexes. Also included is an account of acyclic models for crossed complexes; this is a basic technique for chain complexes in algebraic topology, and the version for crossed complexes has a few twists to make it work.

Chapter 11 deals with the cubical classifying space of a crossed complex. We give this cubical version because it has convenient properties, and also because cubical methods fit better with other techniques of this book, used extensively in Part III. Thus basic results on collapsing in cubical sets are useful for establishing properties of the category of cubical sets, and are also used essentially in Part III for the proof of the Higher Homotopy Seifert–van Kampen Theorem.

In Chapter 12 we begin with the theory of fibrations of crossed complexes and their long exact sequences. These are used with the methods of the classifying space to discuss the homotopy classification of maps of spaces. A notable feature of our methods is that we are able to make some explicit computations, for example of the k-invariants of certain crossed modules. As another sample calculation, we compute in Example 12.3.13 certain homotopy classes of maps from $\mathbb{R}P^2 \times \mathbb{R}P^2$ to the space $\mathbb{R}P^3$ with homotopy groups in dimensions higher than 3 killed. We also relate the crossed complex methods with those of Čech cohomology and with the cohomology of groupoids.

In this way we fulfill the promise of the Introduction to the book, except for the proofs of those results which require the techniques of Part III.