Introduction to Part III

In Part II we have explored the techniques of crossed complexes, and hope we have shown convincingly that they are a powerful tool in algebraic topology. In this part, we give the proofs of the main theorems on which those tools depend.

To this end, we introduce the algebra of ω-groupoids, or in full, *cubical ω-groupoids with connections*. It was the way in which this algebra could be developed to model the geometry of cubes which suggested the possibility of the theory and calculations described in this book.

As intimated in Chapter 6 of Part I, the crucial advantages of cubical methods are the capacity to encode conveniently:

A) subdivision;
B) multiple composition as an algebraic inverse to subdivision;
C) commutative cubes, and their composition.

These properties allow us to prove a HHSvKT by verifying the required universal property: here A) and B) are used to give a candidate for a morphism, and C) is used to verify that this morphism is well defined.

A further advantage of the cubical methods is:

D) the formula $I^m \otimes I^n \cong I^{m+n}$ allows for a convenient modelling of homotopies and higher homotopies.

The techniques which enable analogous arguments for A)–C) in all dimensions are more elaborate than those of Part I. The main achievements are as follows:

- In order to *define* the notion of commutative shell, we have to relate the cubical theory of ω-groupoids to that of crossed complexes. This purely algebraic equivalence is established in Chapter 13, and is a central feature of this book.
- The proof in Chapter 14 that the natural definition of the fundamental ω-groupoid ρX_* of a filtered space actually is an ω-groupoid requires the techniques of *collapsing* for subcomplexes of a cube which were given in Chapter 11. These techniques are also used to prove the equivalence of the two functors ρ and Π under the equivalence of algebraic categories proved in Chapter 13.
- The proof of the HHSvKT for the functor ρ is also given in Chapter 14.
- The penultimate Chapter 15 constructs the monoidal closed structure on the category of ω-groupoids, and deduces the precise formulae for the equivalent structure on crossed complexes used in Part II. Also proved is the Eilenberg–Zilber type natural transformation $\rho(X_*) \otimes \rho(Y_*) \to \rho(X_* \otimes Y_*)$, for filtered spaces X_*, Y_*.
- The final Chapter 16 points to a number of areas which require further study and research to develop further this new foundation for algebraic topology.