Introduction

Supersymmetry (SUSY) is the machinery mathematicians and physicists have developed to treat two types of elementary particles, *bosons* and *fermions*, on the same footing. Supergeometry is the geometric basis for supersymmetry; it was first discovered and studied by physicists, Wess and Zumino [80], Salam and Strathdee [65] (among others), in the early 1970s. Today supergeometry plays an important role in high energy physics. The objects in super geometry generalize the concept of smooth manifolds and algebraic schemes to include anticommuting coordinates. As a result, we employ the techniques from algebraic geometry to study such objects, namely A. Grothendieck's theory of schemes.

Fermions include all of the material world; they are the building blocks of atoms. Fermions do not like each other. This is in essence the Pauli exclusion principle which states that two electrons cannot occupy the same quantum mechanical state at the same time. Bosons, on the other hand, can occupy the same state at the same time.

Instead of looking at equations that simply describe either bosons or fermions separately, supersymmetry seeks out a description of both simultaneously. Transitions between fermions and bosons require that we allow transformations between the commuting and anticommuting coordinates. Such transitions are called supersymmetries.

In classical Minkowski space, physicists classify elementary particles by their mass and spin. Einstein's special theory of relativity requires that physical theories must be invariant under the Poincaré group. Since observable operators (e.g. Hamiltonians) must commute with this action, the classification corresponds to finding unitary representations of the Poincaré group. In the SUSY world, this means that mathematicians are interested in unitary representations of the super Poincaré group. A "super" representation gives a "multiplet" of ordinary particles which include both fermions and bosons.

Up to this point, there have been no colliders that can produce the energy required to physically expose supersymmetry. However, the Large Hadron Collider (LHC) in CERN (Geneva, Switzerland) became operational in 2007. Physicists are planning proton–proton and proton–antiproton collisions which will produce energies high enough where it is believed supersymmetry can be seen. Such a discovery will solidify supersymmetry as the most viable path to a unified theory of all known forces. Even before the boson–fermion symmetry which SUSY presupposes is proved to be physical fact, the mathematics behind the theory is quite remarkable. The concept that space is an object built out of local pieces with specific local descriptions has evolved through many centuries of mathematical thought. Euclidean and non-Euclidean geometry, Riemann surfaces, differentiable manifolds, complex manifolds, algebraic varieties, and so on represent various stages of this concept. In Alexander Grothendieck's theory of schemes, we find a single structure that encompasses all previous ideas of space. How-

ever, the fact that conventional descriptions of space will fail at very small distances (Planck length) has been the driving force behind the discoveries of unconventional models of space that are rich enough to portray the quantum fluctuations of space at these unimaginably small distances. Supergeometry is perhaps the most highly developed of these theories; it provides a surprising application and continuation of the Grothendieck theory and opens up large vistas. One should not think of it as a mere generalization of classical geometry, but as a deep continuation of the idea of space and its geometric structure.

Out of the first supergeometric objects constructed by the pioneering physicists came mathematical models of superanalysis and supermanifolds independently by F. A. Berezin [10], B. Kostant [49], D. A. Leites [53], and De Witt [25]. The idea to treat a supermanifold as a ringed space with a sheaf of $\mathbb{Z}/2\mathbb{Z}$-graded algebras was introduced in these early works. Later, Bernstein [22] and Leites [53] used techniques from algebraic geometry to deepen the study of supersymmetry. In particular, Bernstein and Leites accented the functor of points approach from Grothendieck's theory of schemes. Interest in SUSY has grown in the past decade, and most recently works by V. S. Varadarajan [76] and others have continued exploration of this beautiful area of physics and mathematics and have inspired this work. Given the interest and the number of people who have contributed greatly to this field from various perspectives, it is impossible to give a fair and accurate account of all the works related to ours. We have nevertheless made an attempt and have provided bibliographical references at the end of each chapter, pointing out the main papers that have inspired our work. We apologize for any involuntary omissions.

In our exposition of mathematical SUSY, we use the language of T-points to build supermanifolds up from their foundations in $\mathbb{Z}/2\mathbb{Z}$-graded linear algebra (superalgebra). The following is a brief description of our work.

In Chapter 1 we begin by studying $\mathbb{Z}/2\mathbb{Z}$-graded linear objects. We define super vector spaces and superalgebras, then generalize some classical results and ideas from linear algebra to the super setting. For example, we define a super Lie algebra, discuss supermatrices, and formulate the super trace and determinant (the Berezinian). We also discuss the Poincaré–Birkhoff–Witt theorem in full detail.

In Chapter 2 we provide a brief account of classical sheaf theory with a section dedicated to schemes. This is meant to be an introductory chapter on this subject and the advanced reader may very well skip it.

In Chapter 3 we introduce the most basic geometric structure: a superspace. We present some general properties of superspaces which lead into two key examples of superspaces, supermanifolds and superschemes. Here we also introduce the notion of T-points which allows us to treat our geometric objects as functors; it is a fundamental tool to gain geometric intuition in supergeometry.

Chapters 4–9 lay down the full foundations of C^∞-supermanifolds over $\mathbb{R}$. In Chapter 4, we give a complete proof of foundational results like the chart theorem and the correspondence between morphisms of supermanifolds and morphisms of the superalgebras of their global sections. In Chapter 5 we discuss the local structure

of morphisms proving the analog of the inverse function, submersion and immersion theorems. In Chapter 6 we prove the local and global Frobenius theorem on super-manifolds. In Chapters 7 and 8 we give special attention to super Lie groups and their associated Lie algebras, as well as look at how group actions translate infinitesimally. We then use infinitesimal actions and their characterizations to build the super Lie sub-group, subalgebra correspondence. Finally in Chapter 9 we discuss quotients of Lie supergroups.

Chapters 10, 11 expand upon the notion of a superscheme which we introduce in Chapter 3. We immediately adopt the language of T-points and give criteria for representability: in supersymmetry it is often most convenient to describe an object functorially, and then show that it is representable. We explicitly construct the Grassmannian functorially, then use the representability criterion to show that it is a superscheme. Chapter 10 concludes with an examination of the infinitesimal theory of superschemes.

We continue this exploration in Chapter 11 from the point of view of algebraic supergroups and their Lie algebras. We discuss the linear representations of affine algebraic supergroups; in particular we show that all affine supergroups are realized as subgroups of the general linear supergroup.

We have made an effort to make this work self-contained and suggest that the reader begins with Chapters 1–3, but Chapters 4–9 and Chapters 10–11 are somewhat disjoint and may be read independently.