1 Introduction

This research monograph is intended as an introduction to, and exposition of, some of the phenomena that solutions of nonlinear dispersive focusing equations exhibit at energy levels strictly above that of the ground state soliton. It grew out of lectures that the authors have given on various aspects of their work on focusing wave equations. In particular, it is a much expanded version of the second author's postgraduate course (Nachdiplomvorlesung), which he taught in the fall of 2010 at ETH Zürich, Switzerland.

The equations which we consider in these lectures are Hamiltonian, but not completely integrable, and they all exhibit soliton-like (i.e., stationary or periodic) solutions which are unstable. Amongst those the ground state is singled out as the one of smallest energy. The aforementioned phenomena concern the *transition* from a region of phase space in which solutions exist globally in forward time and scatter to a free wave, to one where they blow up in finite positive time. We can describe this transition in some detail provided the energy is only slightly larger than that of the ground state. In fact, the boundary along which these open regions meet can be identified as a *center-stable manifold* associated with the ground state.

To be more specific, consider an energy subcritical Klein–Gordon equation

$$\ddot{u} - \Delta u + u = f(u) \tag{1.1}$$

in $\mathbb{R}_t \times \mathbb{R}^d_x$ with real-valued solutions. More generally, the *mass term* should be $m^2 u$ with $m > 0$, but we can set $m = 1$ without loss of generality. This equation should be thought of as perhaps the simplest model equation which exhibits the phenomena which we wish to describe here, but it should not be mistaken as the central object of our investigations. We begin with a fairly general discussion of (1.1). It is invariant under the full Poincaré group, i.e., under the group generated by spatial as well as temporal translations, Euclidean rotations, and Lorentz transforms. The latter are defined as the group that leave the quadratic form $\tau^2 - |\xi|^2$ in $\mathbb{R}^{d+1}$ invariant (the Minkowski metric).

Moreover, (1.1) is both Lagrangian as well as Hamiltonian in the following sense: at least formally, solutions to this equation are characterized as critical point of the Lagrangian, with $F' = f$,

$$\mathfrak{L}(u, \dot{u}) = \int_{\mathbb{R}^{d+1}_{t,x}} \Big[-\frac{1}{2}\dot{u}^2 + \frac{1}{2}|\nabla u|^2 + \frac{1}{2}u^2 - F(u) \Big](t, x)\, dt dx\,.$$

This can be seen by integrating by parts in the integral representing $\mathfrak{L}'(\vec{u}) = 0$. The Lagrangian point of view is important with respect to conservation laws generated by one-parameter subgroups of symmetries of $\mathfrak{L}$ (Noether's theorem). For example, time translation invariance leads to the conservation of the energy

$$E(u, \dot{u}) = \int_{\mathbb{R}^d} \Big[\frac{1}{2}\dot{u}^2 + \frac{1}{2}|\nabla u|^2 + \frac{1}{2}u^2 - F(u)\Big](t, x)\, dx$$

invariance under spatial translations yields the conservation of the momentum $P(u) = \langle \dot{u} | \nabla u \rangle$. Euclidean rotations are associated with the conservation of the angular momentum.

The aforementioned energy subcriticality assumption which we made on (1.1) now means the following: the nonlinear term $F\big(u(t)\big)$ is strictly weaker than the H^1 part of the energy, as expressed by the Sobolev estimate. To be more specific, consider $f(u) = \lambda |u|^{p-1} u$ in $\mathbb{R}^3$. Then $p < 5$ is subcritical, whereas $p = 5$ is critical and $p > 5$ is supercritical due to the Sobolev embedding $H^1(\mathbb{R}^3) \subset L^6(\mathbb{R}^3)$. We shall not touch the supercritical case here at all. Even though it may seem most desirable to restrict one's attention to classical, i.e., smooth, solutions of (1.1) this is not the case; the best notion of solution for many different reasons turns out to be that of an *energy solution* which is a solution which belongs to $H^1 \times L^2$ for all times.

To express (1.1) in Hamiltonian form, we write it as a first order system, with dependent variable $U := \binom{u}{\dot{u}}$:

$$\dot{U} = JHU + N(U)$$

where

$$J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad H = \begin{pmatrix} -\Delta + 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad N(U) = \begin{pmatrix} 0 \\ f(u) \end{pmatrix}.$$

For simplicity, let $f = 0$. Then energy conservation simply means that $\frac{d}{dt}\langle HU | U \rangle = 0$. The symplectic form associated with (1.1) can be now seen to be

$$\omega(U, V) = \big\langle JU | V \big\rangle = \int_{\mathbb{R}^d} (U_2 V_1 - U_1 V_2)(x)\, dx\,.$$

There are two main classes dividing equations of the form (1.1): the *defocusing* equations on the one hand, and the *focusing* ones on the other hand.

Loosely speaking, this division can be expressed along the lines of the global-in-time existence problem for (1.1) for smooth, compactly supported data, say. Defocusing equations admit smooth solutions for all data and all times, whereas the focusing ones may exhibit finite time blowup for certain data (such as those of negative energies). To be more specific, consider monic nonlinearities $f(u) = \lambda|u|^{p-1}u$ in arbitrary dimensions. Then $\lambda > 0$ represents the focusing case, and $\lambda < 0$ the defocusing one. Note that this corresponds exactly to the distinction of the energy $E(\vec{u})$ being indefinite vs. positive definite, respectively.

Only the focusing case will be relevant to this monograph. Of central importance to the theory of the focusing nonlinear Klein–Gordon (NLKG) equation (1.1) is the fact that they admit nonzero time-independent solutions φ. Any weak H^1 solution of the semilinear elliptic PDE

$$-\Delta\varphi + \varphi = f(\varphi) \tag{1.2}$$

is such a solution. Letting the Poincaré symmetries act on φ generates a manifold of moving solutions of the following form: first, define for any $(p, q) \in \mathbb{R}^{2d}$

$$Q(p,q)(x) = Q(x - q + p(\langle p\rangle - 1)|p|^{-2} p \cdot (x - q)),$$

where $\langle p\rangle := \sqrt{1 + |p|^2}$. The traveling waves generated from φ are defined as

$$u(t) = \pm\varphi(p, q(t)), \quad p \in \mathbb{R}^d,\ \dot{q}(t) = \frac{p}{\langle p\rangle},$$

with fixed momentum p and velocity $\frac{p}{\langle p\rangle}$. They are solutions of (1.1). Note that $|\dot{q}(t)| < 1$ in agreement with the fact that the speed of light which is normalized to equal 1, acts as a barrier.

Amongst all solutions of (1.2) one singles out a positive decaying one, called the *ground state* which we denote by Q. It is known to be unique up to translations for many different nonlinearities $f(u)$, and it is radial. Q is characterized as the minimizer of the stationary energy (or action)

$$J(\varphi) := \int_{\mathbb{R}^3} \left[\frac{1}{2}|\nabla\varphi|^2 + \frac{1}{2}\varphi^2 - F(\varphi)\right] dx$$

subject to the constraint, with $\varphi \neq 0$,

$$K_0(\varphi) := \int_{\mathbb{R}^3} \left[|\nabla\varphi|^2 + \varphi^2 - f(\varphi)\varphi\right] dx = 0 \tag{1.3}$$

It follows that the regions

$$\begin{aligned} \mathcal{PS}_+ &= \{(u, \dot{u}) \mid E(u, \dot{u}) < J(Q),\ K_0(u) \geq 0\}, \\ \mathcal{PS}_- &= \{(u, \dot{u}) \mid E(u, \dot{u}) < J(Q),\ K_0(u) < 0\} \end{aligned} \tag{1.4}$$

are invariant under the nonlinear flow in the phase space $H^1 \times L^2$ where $E(u, \dot{u})$ is the conserved energy for (1.1). It is a classical result of Payne and Sattinger [114] that solutions in $\mathcal{PS}_+$ are global, whereas those in $\mathcal{PS}_-$ blow up in finite time; these results apply to *both time directions*, i.e., blowup occurs for both positive and negative times simultaneously, and the same is true of global existence. In particular, the stationary solution Q is unstable, see also Shatah [125] and Berestycki, Cazenave [11]. Scattering in $\mathcal{PS}_+$ was only recently shown by Ibrahim, Masmoudi, and the first author [77] using the concentration-compactness proof method of Kenig and Merle [84]. We present these results below the energy threshold $J(Q)$ in Chapter 2.

Starting with Chapter 3 we study solutions whose energies satisfy

$$J(Q) \leq E(u, \dot{u}) < J(Q) + \varepsilon^2 \tag{1.5}$$

for some small $\varepsilon > 0$ and the special nonlinearity $f(u) = u^3$ (although this is out of convenience rather than necessity). It is here that one encounters the aforementioned center-stable manifolds that appear as boundaries of open blowup/global existence regions.

Center/stable/unstable manifolds are well-established objects which arise in the study of the asymptotic behavior of ODEs in $\mathbb{R}^n$, see for example Carr [25], Hirsch, Pugh, Shub [72], as well as Guckenheimer, Holmes [68], and Vanderbauwhede [138].

Let us recall the meaning of these manifolds: given an ODE in $\mathbb{R}^n$, $\dot{x} = f(x)$ with $f(0) = 0$, and f smooth, let $A = Df(0)$. Then split $\mathbb{R}^n = X_u + X_s + X_c$ as a direct sum into A-invariant subspaces such that all eigenvalues of $A \restriction X_u$ lie in the right half-plane, those of $A \restriction X_s$ lie in the left half-plane, and the eigenvalues of $A \restriction X_c$ are all purely imaginary. An example of such a situation is given by the 7×7 matrix

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}. \tag{1.6}$$

The eigenvalues are $\{0, 1, -1, i, -i\}$, and X_u is spanned by the first two coordinate directions, X_s by the third, and the center subspace by the final four. Note that X_c splits into a rotation in the fourth and fifth variables, but has linear growth on the subspace spanned by variables six and seven. By the *center-manifold* theorem, see [25], [68], [138], there exist smooth manifolds M_u, M_s, M_c locally around zero which are tangent to X_u, X_s, X_c, respectively, at $x = 0$, and which are transverse to each other. Moreover, M_u, M_s, M_c are each locally invariant under the flow (meaning that a trajectory starting on any of these, say M_c, remains on M_c as long as the trajectory itself remains in a small neighborhood of the equilibrium point). A manifold M_{cs} which is tangent to $X_s + X_c$ at $x = 0$, of the same dimension as this tangent space, and is locally invariant under the flow is referred to as *center-stable*. On M_s, M_u the solution to $\dot{x} = f(x)$ decreases exponentially fast as $t \to \infty$ or $t \to -\infty$, respectively (in fact, they are characterized by this property). But on M_c the behavior can be quite complicated and that manifold is not characterized by growth conditions.

Such a decomposition is relevant for several reasons. On the one hand, it reduces the dynamics in the state space to lower-dimensional subspaces which is often the only way to obtain any understanding of the flow. On the other hand, it is most relevant for bifurcation theory of ODEs which refers to situations where the vector field $f(x)$ depends on a parameter μ, see Guckenheimer, Holmes [68].

In the context of (1.1) and related Hamiltonian PDEs such as the cubic focusing nonlinear Schrödinger equation (NLS), a center-stable manifold arose in [122] from the attempt of obtaining a *conditional asymptotic stability* result for an unstable equation. More precisely, the second author obtained – for the cubic focusing NLS in $\mathbb{R}^3$, and in a small neighborhood of the ground state soliton – a codimension one manifold with the property that any solution starting from that manifold exists globally and scatters to a (modulated) ground state soliton. The drawback of [122] lay with the topology which is not invariant under the NLS flow. But Beceanu [9] later carried out the construction in the optimal topology introducing several novel ideas, such as Strichartz estimates for linear evolution equations with small time-dependent but space-independent lower order terms. This allowed him not only to obtain a similar conditional asymptotic stability result as the one in [122] (but of course without any pointwise control on the rate of convergence of various parameters, which requires a stronger topology), but also to verify the properties usually associated with the center-stable manifold such as invariance locally in time (in fact, globally in forward time, and locally in backward time).

The method of proof in [122], [9] is perturbative, and is restricted to a small neighborhood of the ground states. This work left open the question as to what happens near the ground state soliton, but off the center-stable manifold. This is one of the problems we wish to address in this monograph. While there has been some heuristic and numerical work in the physics literature, see Bizoń et al. [15], [16] and Choptuik [30], the first rigorous results on this problem were obtained in

[87], [88], [109]–[111]. As an example, consider the one-dimensional Klein–Gordon equation

$$u_{tt} - u_{xx} + u = |u|^{p-1}u \tag{1.7}$$

with $p > 5$. It is well-known that for this equation the solitons are given by the explicit expressions

$$Q(x) = \alpha \cosh^{-\frac{1}{\beta}}(\beta x), \quad \alpha = \left(\frac{p+1}{2}\right)^{\frac{1}{p-1}}, \ \beta = \frac{p-1}{2}.$$

In contrast to the nonlinear Schrödinger equation, one of the advantages of (1.7) lies with the fact that under an *even perturbation* Q does not change; in other words, it is not modulated. In fact, the following result was obtained in [88].

Theorem 1.1. *Let $p > 5$. There exists $\varepsilon > 0$ such that any even real-valued data $(u_0, u_1) \in H^1(\mathbb{R}) \times L^2(\mathbb{R})$ with energy*

$$E(u, \dot{u}) < E(Q, 0) + \varepsilon^2 \tag{1.8}$$

have the property that the solutions $u(t)$ of (1.7) associated with these data exhibit exactly one scenario of the following trichotomy:

- *u blows up in finite positive time*
- *u exists globally and scatters to zero as $t \to \infty$*
- *u exists globally and scatters to Q, i.e., there exists a free Klein–Gordon wave $\big(v(t), \dot{v}(t)\big) \in H^1 \times L^2$ with the property that*

$$\big(u(t), \dot{u}(t)\big) = (Q, 0) + \big(v(t), \dot{v}(t)\big) + o_{\mathcal{H}}(1), \quad t \to \infty.$$

In addition, the set of even data as above splits into nine nonempty disjoint sets corresponding to all possible combinations of this trichotomy in both forward and negative times.

All solutions which fall under the third alternative form the center-stable manifold. One can show that this is a C^1 (or better) manifold of codimension 1 which passes through $(Q, 0)$ and lies in the set described by (1.8). Figure 1.1 illustrates Theorem 1.1 for data that start off near the ground-state solution $(Q, 0)$. The third region is the *center-stable* manifold. The figure on the cover illustrates the nine sets alluded to in the final statement of the theorem.

Moreover, we obtain the following characterization of the threshold solutions, i.e., those with energies $E(\vec{u}) = E(Q, 0)$. Results of this type originate with the seminal work of Duyckaerts and Merle [48].

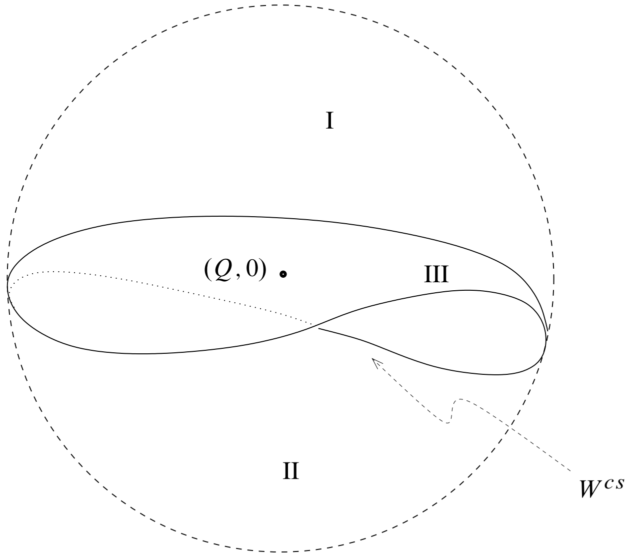

Figure 1.1. The forward trichotomy near $(Q, 0)$

Corollary 1.2. *The even solutions to* (1.7) *with energy* $E(\vec{u}) = E(Q, 0)$ *are characterized by exactly one of the following scenarios, each of which can occur:*

- *they blow up in both the positive and negative time directions*
- *they exist globally on* $\mathbb{R}$ *and scatter as* $t \to \pm\infty$
- *they are constant and equal* $\pm Q$
- *they equal one of the following solutions, for some* $t_0 \in \mathbb{R}$*:*

$$\sigma W_+(t + t_0, x), \quad \sigma W_-(t + t_0, x), \quad \sigma W_+(-t + t_0, x), \quad \sigma W_-(-t + t_0, x)$$

where $(W_\pm(t, \cdot), \partial_t W_\pm(t, \cdot))$ *approach* $(Q, 0)$ *exponentially fast in* $\mathcal{H}$ *as* $t \to \infty$*, and* $\sigma = \pm 1$*. In backward time,* W_+ *scatters to zero, whereas* W_- *blows up in finite time.*

As usual, the images of $W_\pm$ and Q form the one-dimensional *stable manifold* associated with $(Q, 0)$. The *unstable manifold* is obtained by time-reversal. The goal of these lectures is to prove results such as Theorem 1.1 and Corollary 1.2. More precisely, in Chapters 2–5 we restrict ourselves to the radial cubic NLKG equation in $\mathbb{R}^3$ and systematically develop the machinery leading to results analogous to Theorem 1.1 and Corollary 1.2 above.

Loosely speaking, the argument relies on an interplay between the hyperbolic dynamics near the ground states on the one hand, and a variational analysis away from them on the other hand. While the former here means linearizing around the ground states and then performing the necessary perturbative analysis locally around the ground states, the latter refers to a type of global argument that is not at all based on linearization. More precisely, we shall use the virial identity valid for any energy solution of (1.1),

$$\frac{d}{dt}\Big\langle w\dot{u} \mid \frac{1}{2}(x\cdot\nabla+\nabla\cdot x)u\Big\rangle = -K_2(u) + \text{error} \tag{1.9}$$

where $K_2(u) = \|\nabla u\|_2^2 - \frac{3}{4}\|u\|_4^4$. The cut-off function w here is chosen in such a way that the error remains small, cf. Figure 4.6 on page 165. From variational considerations we shall be able to conclude that $K_2(u)$ has a definite sign away from a neighborhood of $\pm Q$, and by integration of (1.9) we will obtain the important no-return (or "one-pass") theorem, see Chapter 4. This guarantees that solutions which are not trapped by $\pm Q$ can only make one pass near the ground states which then implies that the signs of $K_0(u), K_2(u)$ stabilize. The latter then allows one to conclude finite time blowup or global existence in the same way as Payne and Sattinger [114], at least for those solutions which are not trapped by the ground states. Those that are trapped are then shown to lie on the center-stable manifold whence the third alternative in Theorem 1.1.

Since very little is known about solutions in the regime $E(u,\dot{u}) > J(Q)+\varepsilon^2$, it seems natural to turn to numerical investigations in order to obtain some idea of the nature of the blowup/global existence dichotomy. Roland Donninger and the second author have conducted such computer experiments at the University of Chicago, see [45]. This work consists of numerical computations of radial solutions to (1.1) with $f(u) = u^3$ whose data belong to a two-dimensional surface (such as a planar rectangle) in the infinite dimensional phase space $\mathcal{H} := H^1 \times L^2$ (of course the data are chosen to belong to a fine rectangular grid on that surface). Each solution is then evaluated with regard to blowup/global existence and a dot is placed on the data rectangle if global existence is observed, whereas the dot is left blank otherwise. Figure 1.2 below shows the outcome of such a computation for the data choice

$$\big(u(0),\dot{u}(0)\big)(r) = \big(Q(r) + Ae^{-r^2}, Be^{-r^2}\big)$$

with the horizontal axis being A, and the vertical being B. The central region, from which thin spikes emanate, is the set of data leading to global existence. The drop-shaped region contained inside of it is the set $\mathcal{PS}_+$, see (1.4), whereas both the region to the far right (which meets $\mathcal{PS}_+$ at a cusp centered at $(Q,0)$ which corresponds to $A = B = 0$) as well as the region on the far left are $\mathcal{PS}_-$. The region which appears blank is the one giving finite time blowup (at least numerically), and it contains $\mathcal{PS}_-$ as a subset.

We refer the reader to [45] for a discussion of the numerical methods, as well as more results and figures. The appearance of the two Payne–Sattinger regions near the point $(0,0)$ is reminiscent of the set $\xi^2 - \eta^2 \leq 0$. This is due to the fact that the energy near Q takes the form of a saddle surface, which in turn follows from the existence of negative spectrum of the linearized operator $L_+ = -\Delta + 1 - 3Q^2$, see Section 3.1. In fact, there is a codimension-1 plane around $(Q, 0)$ in $\mathcal{H}$ such that locally around that point the energy is positive definite on this plane, whereas it is indefinite on the whole space. An important feature of the central global existence region in Figure 1.2 is the appearance of the boundary: it seems to be a smooth curve. In fact, we will prove in Chapter 3 that *near* $(Q, 0)$ in $\mathcal{H}$ the boundary is indeed a smooth codimension 1 manifold $\mathfrak{M}$ with the property that solutions with data on that manifold are global and scatter to Q as $t \to \infty$. In dynamical terms this manifold is precisely the *center-stable* one, which contains the 1-dimensional stable manifold. Furthermore, $\mathfrak{M}$ is transverse to the 1-dimensional unstable manifold. The latter manifold is characterized by the property that all solutions starting on it converge to $(Q, 0)$ as $t \to -\infty$; in fact, this convergence is exponential. Moreover, in positive times solutions on the unique unstable manifold grow exponentially up until the time at which they leave a small neighborhood of the equilibrium $(Q, 0)$.

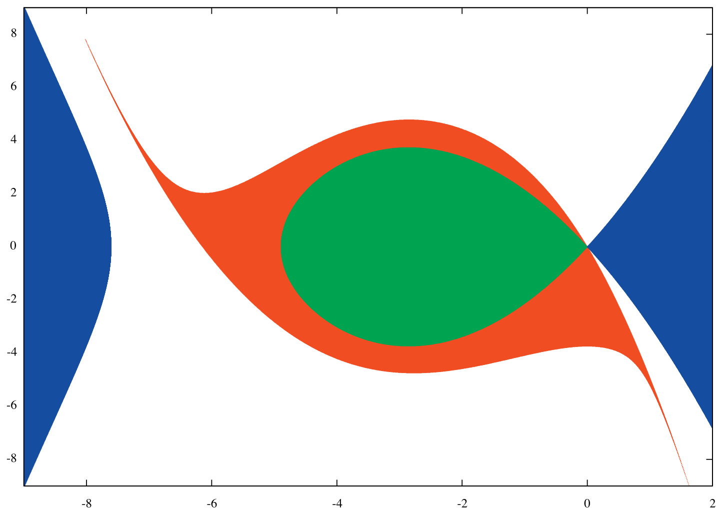

Figure 1.2. Numerically computed planar section through $(Q, 0)$

In their ongoing numerical investigations, Donninger and the second author have found that the boundary of the forward scattering region, i.e., of

$$\mathcal{S}_+ := \{(u(0), \dot{u}(0)) \mid u \text{ exists for all positive times and scatters to zero}\}$$

is more complicated than one might expect; more specifically, numerical experiments suggest that the boundary can display features atypical of a smooth manifold, such as many thin filaments which emerge from it, see [45] for a precise description of this phenomenon. However, the numerics in and of itself does not provide any conclusive evidence at this point that could indicate that the boundary is *not* a smooth manifold.

The results in this book are based on the following papers: in [109], [111] the authors studied the radial as well as nonradial nonlinear cubic Klein–Gordon equation in $\mathbb{R}^3$, and in [110] they treated the radial nonlinear focusing cubic Schrödinger equation in $\mathbb{R}^3$. The so-called graph transform (also known as Hadamard's method) is adapted to the nonradial NLKG equation in [112] in order to construct invariant manifolds. The point of [112] is to be able to separate the issue of the *construction* of the invariant manifolds from that of establishing *asymptotic properties of those solutions which start on the center-stable manifold*. While relatively "soft" spectral information suffices for the former, the latter – at least with current technology – depends on "hard" information such as determining the spectrum in the gap of the linearized operator and understanding the resolvent at the threshold, i.e., the edge of the continuous spectrum.

To be specific, the graph transform on which [112] is based requires no knowledge of the spectrum of linearized operator in the gap, whereas the Lyapunov–Perron method in the implementation of [109], [111] does demand such spectral information. On the other hand, the graph transform does not allow any conclusions about the asymptotic stability properties for solutions belonging to the center-stable manifold. See Chapter 3 for a discussion of both methods. The energy critical wave equation in $\mathbb{R}^3$ and $\mathbb{R}^5$ was studied in [87] by J. Krieger and the authors, whereas the one-dimensional nonlinear Klein–Gordon equation was treated by these same authors in [88].

The book is organized as follows. Chapters 2 through 5 are largely devoted to the cubic focusing Klein–Gordon equation in $\mathbb{R}^3$, mostly under a radial assumption (one exception being the nonradial scattering result from [77] which applies to energies *below* that of the ground state). The NLS equation appears in Chapter 3 where we establish the existence of center-stable manifolds for the cubic equation in $\mathbb{R}^3$. The cubic NLKG equation in three dimensions turns out to be particularly well-suited for the development of the new results above the ground state energy, since it exhibits both finite propagation speed as well as a lack of symmetries (due to the mass term), at least in the radial context. The latter is convenient as it implies that stationary solutions remain static and unchanged under small perturbations (more technically speaking, one does not need any modulation parameters).

In Chapter 2, we review the basics of the theory, including local and global well-posedness of the equation, as well as the Payne–Sattinger results characterizing global existence/finite time blowup below the ground state energy. Most of the effort goes into an exposition of the scattering results from [77], which rely on the Kenig–Merle method [84] and the concentration-compactness decomposition as in Bahouri, Gérard [4], and Merle, Vega [105]. The origins of this decomposition go back to the ideas introduced by Lions [98] in the elliptic context, but the adaptation to Hamiltonian evolution equations is highly nontrivial.

For pedagogical reasons, we split the scattering proof into the radial and nonradial cases, respectively, with the former of course being simpler. Due to the subcriticality of the problem, the Kenig–Merle method is easier to implement for the cubic NLKG equation than for the critical wave equation as considered in [84]. In order to keep our presentation largely self-contained, Chapter 2 concludes with a presentation of the Strichartz estimates for Klein–Gordon equations.

Chapter 3 initiates the discussion of the dynamics for energies which are slightly larger than the ground state energy. More specifically, we first introduce the concept of stable/unstable/center manifolds by means of the work of Bates and Jones [6], followed by a discussion of the work by the second author [122], and Beceanu [9] on the (cubic) NLKG and NLS equations. We also review some of the linear dispersive theory needed in that context. Technically speaking, [6] adapts the Hadamard method (of invariant cones) to the context of ODEs in Banach spaces, which is then applied to the NLKG equation. On the other hand, [122], as well as [9], [87], [90], [91], [109]–[111] use the Lyapunov–Perron method which allows for a detailed description of the *asymptotic dynamics* of solutions starting from the center-stable manifold. The "cost" of this lies with Strichartz estimates for the linearized operator which in turn rely on a careful spectral analysis of that operator. While the Hadamard method yields less information, it also requires much less information on the linearized evolution. We wish to emphasize, though, that any sophisticated spectral information such as the *gap property* studied in [41] is needed only for the *scattering property* of solutions lying in the center-stable manifold but not for the construction of the invariant manifolds themselves. See [112] for more on this matter.

In Chapter 4 we present the core of the method developed in [109]. Loosely speaking, one combines the unstable hyperbolic dynamics near the ground states with the variational structure of the stationary energy functional J, as well as of functionals derived from J via the action of symmetries on the equation (for example, the Payne–Sattinger functional K_0 or the virial functional K_2, see above). Of particular importance in Chapter 4 is the *one-pass theorem* which says that there do not exist almost homoclinic orbits connecting $\{(\pm Q,0)\}$ with itself (more precisely, connecting small balls around $(\pm Q,0)$).

The analysis of Chapters 3 and 4 then leads to a description of the global dynamics analogous to Theorem 1.1 above. This is carried out in Chapter 5, which ends with a summary of the methods developed here. In particular, we obtain the 9-set theorem for the cubic NLKG equation in $\mathbb{R}^3$. We again use the Kenig–Merle method to prove scattering to zero, but the execution of this method relies crucially on the aforementioned one-pass theorem.

The final Chapter 6 presents other results which are accessible (but with considerable additional work in most cases) to the ideas set forth in this book. More specifically, we consider the nonradial form of (1.1), the one-dimensional NLKG with even data, the cubic radial NLS equation in $\mathbb{R}^3$, and finally the energy critical wave equation in dimensions 3 and 5. In the critical case, our results are less complete than they are in the subcritical case. In contrast to the previous chapters, the final one is purely expository and presents only select details. It is meant as an introduction to the original research presented in [87], [88], [110], [111].

There are several exercises throughout the text, most of which appear in Chapters 2 and 3. The starred ones are somewhat more involved. Some exercises ask the reader to supply technical details that were omitted from an argument. Those exercises should be considered part of the main body of the text. Exercises marked with a dagger † go beyond the core material, and are thus not needed in order to follow the proofs. For the most part, those exercises tend to be quite involved as well.

The questions and problems addressed in this monograph can be seen from several perspectives: (i) On the one hand, we consider dispersive Hamiltonian equations which in and of themselves constitute a vast and rapidly developing field. As far as the *defocusing case* is concerned, the energy subcritical as well as the energy critical equations have been studied extensively for both wave and Schrödinger equations. The earliest treatment of the global existence problem for semilinear wave equation with smooth solutions was conducted by Jörgens [80] for subcritical defocusing nonlinearities in $\mathbb{R}^3$, and the global problem for the critical case (u^5 nonlinearity in $\mathbb{R}^3$) was solved by Struwe [134] radially, and Grillakis [65] nonradially. The corresponding problem for the critical nonlinear Schrödinger equation in $\mathbb{R}^3$ was settled by Bourgain [19] radially (see also Grillakis [66]), and Colliander, Keel, Staffilani, Takaoka, Tao [37] nonradially. For more on defocusing equations, see the monographs by Bourgain [18] and Tao [136]. An alternative as well as very general approach to global existence problems was found recently by Kenig and

Merle [84]. It applies to both defocusing (see [86]) as well as focusing equations, but for the latter [84] only allows *energies strictly below the ground state energy*. In fact, this entire monograph is devoted to the question of what happens at energies equal to or larger than the ground state energy. On the other hand, Duyckaerts and Merle [48] carried out a comprehensive analysis of the *threshold behavior*, i.e., at energies equal to that of the ground states for both the energy critical NLS and nonlinear wave equations. It turns out that the special threshold solutions $W_\pm$ which they found in this context are of a universal nature; for example, Duyckaerts and Roudenko [49] established their existence for the cubic (and thus energy subcritical) NLS in $\mathbb{R}^3$. Furthermore, in this text we shall identify them as one-dimensional *stable* and *unstable* manifolds, respectively, associated with the ground states.

Even though the literature on focusing equations is generally speaking more sparse than for the defocusing ones, certain classes of equations have been studied in great detail. Especially for the L^2-critical focusing NLS equation substantial progress has been made on the very delicate blowup phenomena exhibited at and near the ground state. The L^2 critical equation is special due to its invariance under the *pseudo-conformal transformation*, see for example [27]. Applying this class of transformations to the ground state Q gives rise to a solution blowing up in finite time, and it is unique with this property at exactly the mass of Q, see Merle [101]. Very recently Merle, Raphaël, and Szeftel [104] proved that these solutions are unstable. Prior to that, and more in the spirit of the present work, Bourgain and Wang [20] studied the *conditional stability* of the pseudo-conformal blowup on a submanifold of large codimension, and Krieger and the second author [89] established the existence of a codimension 1 submanifold (albeit with no regularity and in a strong topology) for which these solutions are preserved. The conjecture that the pseudo-conformal should be stable under a codimension 1 condition is due to Galina Perelman [116].

A sweeping analysis of the *stable* blowup regime near the ground state for the L^2-critical case was carried out by Merle and Raphaël [102] in a series of works, preceded by [116] which established the existence of the so-called $\log\log$ blowup regime. In [103] Merle, Raphaël and Szeftel were able to transfer some of the techniques from the critical equation to the slightly L^2-supercritical one and established stable blowup dynamics near the ground state. The L^2-critical instability of the ground state is *algebraic* in nature rather than exponential, and thus very far from the considerations in this paper. We emphasize that the hyperbolic dynamics is an essential feature of the theory presented in this text. Moreover, in contrast to the aforementioned works, we do not pursue the deep question of characterizing the type of blowup when it occurs; rather, we only establish its existence via the usual convexity obstruction, see [62], [94], [114]. For a characterization of the nature of the blowup, see Merle, Zaag [106], as well as Hamza, Zaag [70].

(ii) On the other hand, one can view this text also from the viewpoint of the vast and diverse body of literature in PDEs which either construct or use (un)stable and center manifolds. Let us merely point out some relevant literature (we caution the reader, however, that the following list is by no means exhaustive). First, Ioss, Vanderbauwhede [78], Gallay [56], and Chen, Hale, Tan [29] construct center(-stable) manifolds in an "abstract" infinite-dimensional setting, as do Bates and Jones [6]. Ball [5] also has a construction which he then applies to the beam equation. The most prevalent method used in the literature appears to be that of Lyapunov–Perron, but [6] follows the so-called Hadamard approach which can sometimes be preferable, cf. [112]. An "abstract" implementation of the Lyapunov–Perron method in infinite dimensions can be found in the seminal papers by Chow, Lu [31] and Chow, Lin, Lu [32] who constructed foliations by invariant manifolds. The reader will find an exposition of both types of constructions in Chapter 3. Bates, Lu, and Zeng [7] developed the theory of center manifolds associated with invariant manifolds which are larger than isolated equilibria. See also Chow, Liu, Yi [33].

Applications of center manifolds to PDEs also abound, especially in the dissipative setting. For example, see Carr, Pego [26], Eckmann, Wayne [50], Wayne [139], Collet, Eckmann [36], Bianchini, Bressan [14], and Beck, Wayne [10], just to name a few. Less work seems to have been done on invariant manifolds for conservative equations, perhaps due to the fact that the center manifold becomes dominant in that case. Promislow [118] applies invariant manifold ideas to a dispersive equation, as do Comech, Cuccagna, Pelinovsky [38], and Weder [142]. Tsai and Yau [137] obtained conditional stability results for excited states of a nonlinear Schrödinger equation with a potential. Soffer and Weinstein [129] obtain the following long-term asymptotic result for the same type of equations studied by Tsai and Yau: either the solution approaches the ground state (the generic case), or an excited state (non-generically).

There have also been many applications of invariant manifolds to the equations of fluid dynamics. For a recent review see Wayne [140], as well as Constantin, Foias [39], Constantin, Foias, Nikolaenko, Temam [40], and Gallay, Wayne [57].

Li, McLaughlin, Shatah, and Wiggins [95] constructed homoclinic orbits for a forced-dissipative perturbation of the completely integrable cubic NLS equation on the one-dimensional torus. Their construction involved, amongst many other elements, invariant manifolds. For more background, as well as a discussion of the relevance of this work towards establishing chaotic motion for NLS, see the book by Li, Wiggins [96]. For an expert summary of [95], and an illuminating overview of much other work in this area see the introduction of the Memoirs article by Bates, Lu, and Zeng [7]. For further work on homoclinic orbits via invariant manifolds see Zeng [146] and Shatah, Zeng [127].

However, we emphasize that all aforementioned references are somewhat different from the subject matter of this monograph, as they do not exhibit the role of a center-stable manifold as the locus of transition from data leading to blowup versus data leading to global existence and scattering (in forward time, say).