Preface

In the first half of the 20th century, analysis went from studying smooth functions to nonsmooth ones, introducing such notions as weak solutions, Sobolev spaces and distributions. These concepts were first studied on $\mathbf{R}^n$ and later on manifolds and other smooth objects. Around 1970 came the first step towards analysis on nonsmooth objects, when notions such as maximal functions and Lebesgue points were studied on spaces of homogeneous type, i.e. on quasimetric spaces equipped with doubling measures. This theory can be called zero-order analysis, as no derivatives are used.

Taking partial derivatives is not possible in metric spaces, but in the 1990s there was a need for studying first-order analysis on nonsmooth spaces. Heinonen and Koskela realized that upper gradients could be used as a substitute for the usual gradient. This gave rise to Newtonian spaces, one of several attempts to define Sobolev spaces on metric spaces, and perhaps the most fruitful one. It turned out that the potential theory of p-harmonic functions can be extended to metric spaces through the use of upper gradients.

The nonlinear potential theory of p-harmonic functions on $\mathbf{R}^n$ has been developing since the 1960s and has later been generalized to weighted $\mathbf{R}^n$, Riemannian manifolds, graphs, Heisenberg groups and more general Carnot groups and Carnot–Carathéodory spaces, and other situations. Studying potential theory of p-harmonic functions on metric spaces generalizes and gives a unified treatment of all these cases.

There are primarily five books devoted to nonlinear potential theory, viz. Adams–Hedberg [5], Heinonen–Kilpeläinen–Martio [171], Malý–Ziemer [258], Mizuta [287] and Turesson [342]. They all study potential theory on unweighted ([5], [258] and [287]) or weighted ([171] and [342]) $\mathbf{R}^n$. In [5] and [342] the focus is on higher order potential theory, whereas the main topic in [171] and [258] is the potential theory of p-harmonic functions (on weighted and unweighted $\mathbf{R}^n$, respectively). The main focus in [287] is on Riesz potentials. The topics covered in our book are closest to [171], but there are also some parts in common with [258]. There is also some overlap, especially in Chapter 8, with the book by Giusti [146] (for unweighted $\mathbf{R}^n$).

Let us also mention the survey papers by Martio [266], which has much in common with this book, and by Björn–Björn [47], which is based on an early version of this book. Moreover, the forthcoming monograph Heinonen–Koskela–Shanmugalingam–Tyson [177] has a certain overlap with the first part of this book.

This book consists of two related parts. In the first part we develop the theory of Newtonian (Sobolev) spaces on metric spaces, and in the second part we develop the potential theory associated with p-harmonic functions on metric spaces.

Both the Newtonian and the p-harmonic theories on metric spaces have now reached such a maturity that we think they deserve to be written in book form. So far, both theories are scattered over a large number of different research papers published during the last two decades, with obvious difficulties for those interested in them. In fact, very few of the results in this book are available in book form.

When writing a book, one is faced with many decisions on what to include. Naturally, the choice is influenced by the taste and interests of the authors.

Throughout the book we consider solely real-valued Newtonian spaces. We also restrict ourselves to the theory of p-harmonic functions (defined through upper gradients) on complete doubling metric spaces supporting a Poincaré inequality. We do not cover quasiminimizers, nor the noncomplete theory. Neither do we include results that only hold for Cheeger p-harmonic functions, or only in $\mathbf{R}^n$.

Related topics that could have been covered but were left out include: Differentiability in metric spaces (in particular, we do not prove Cheeger's theorem (Theorem B.6)) and the Poincaré inequalities in the various examples discussed in Section 1.7 and Appendix A. Including these topics would have made the book substantially different, and many of these topics would rather deserve books on their own.

There are also many recent results which we have not been able to cover, nor have we included a proof of Keith–Zhong's theorem, Theorem 4.30. (See however the appendices, notes and remarks for some references and comments on the topics mentioned above.)

This book is reasonably self-contained and we develop both the Newtonian theory and the p-harmonic theory from scratch. Naturally, we cannot develop all the mathematics needed for this book, and we have most often chosen to omit results which are available elsewhere in book form, but sometimes providing a reference. Thus, the reader is assumed to know measure theory and functional analysis. Apart from comparison results between Newtonian spaces and ordinary Sobolev spaces, there is no background needed in Sobolev space theory.

In Chapters 1 and 2 we start developing the theory of upper gradients and Newtonian functions. Here we have collected the theory which works well in general metric spaces, i.e. without any assumptions such as doubling or the validity of a Poincaré inequality.

In Chapter 3 we introduce the doubling condition and study some of its consequences. The reader interested only in the Newtonian theory can safely skip Sections 3.3–3.5. The John–Nirenberg lemma and its consequences in Sections 3.3 and 3.4, will be used only in the proof of the weak Harnack inequality for superminimizers (Theorem 8.10). (More specifically, it is Corollary 3.21 which is used there.) The Gehring lemma in Section 3.5, although important, is not used in this book.

In Chapter 4 we introduce Poincaré inequalities of various types and look at their relations and some consequences. We discuss, in particular, connections with quasiconvexity. In Chapters 5 and 6 we study various properties of Newtonian spaces, which follow from assuming doubling and a p-Poincaré inequality.

Throughout Chapters 1–6 we have taken extra care to see when the proofs are valid for $p = 1$ and with minimal assumptions, in particular when completeness is not needed. However, as our main interest is in the case $p > 1$, we have not dwelled further on the case $p = 1$ when it requires special proofs.

The second part of the book consists of Chapters 7–14. In it we develop the potential theory associated with p-harmonic functions. Already from the beginning we need to use doubling, a p-Poincaré inequality and that $p > 1$. We also assume completeness, and these assumptions are general throughout the second part. We have taken special care to cover the case when X is a bounded metric space.

In Chapters 7 and 8 we study the basic properties of superminimizers and show interior regularity, whereas Chapter 9 is devoted to superharmonic functions.

In Chapter 10 we look closely at the Dirichlet problem, and in Chapter 11 on boundary regularity. In Chapter 12 we consider removable singularities, which are in turn needed to obtain the trichotomy (Theorem 13.2) motivating the study and classification of irregular boundary points in Chapter 13.

Finally, in Chapter 14 we show that open sets can be approximated by regular sets and give some consequences thereof, including another formulation of the Dirichlet problem.

In Appendix A we give various examples of metric spaces satisfying our basic assumptions (completeness, doubling and the validity of a p-Poincaré inequality). In particular, we show that on weighted $\mathbf{R}^n$ our theory coincides with the one developed in Heinonen–Kilpeläinen–Martio [171] and other sources. When specialized to weighted $\mathbf{R}^n$, many of the results in this book appear in [171]. We have refrained from pointing out exactly which ones in the notes at the end of each chapter. Instead, we make [171] as a general reference for comparison throughout the book. The reader is also referred to the comments and references in [171]. In any case, we would like to point out that many of the results in Chapters 10–14 do not appear in [171] (when specialized to weighted $\mathbf{R}^n$).

In Appendix B we take a quick look at Hajłasz–Sobolev and Cheeger–Sobolev spaces, and in Appendix C we give a short overview of the more general potential theory of quasiminimizers.

The reader interested in open problems should observe that the item *open problem* in the index gives references to all open problems stated in the book.

We started writing this book when giving a graduate course on this topic during the autumn of 2005 in Linköping. In fact, we gave a similar course in Prague in the autumn of 2003, and the handwritten notes we then obtained are in a sense the very first draft of this book. We thank the participants for their active role in the courses: Jan Kališ, Jan Malý, Petr Přecechtěl and Jiří Spurný in Prague, and Gunnar Aronsson, Thomas Bäckdahl, Daniel Carlsson, David Färm, Thomas Karlsson, Mats Neymark, Björn Textorius, Johan Thim and Bengt-Ove Turesson in Linköping.

Later on we have also given lecture series on these topics: Anders at the Paseky spring school *Function Spaces, Inequalities and Interpolation*, in June 2009, on Newtonian spaces, and Jana within the *Taft Research Seminar* at the University of Cincinnati, during the winter term 2010. Again, we thank all the participants.

We are also grateful to Zohra Farnana, Heikki Hakkarainen, Stanislav Hencl, David Herron, Tero Kilpeläinen, Juha Kinnunen, Riikka Korte, Visa Latvala, Tero

Mäkäläinen, Lukáš Malý, Niko Marola, Olli Martio, David Minda, Mikko Parviainen, Nageswari Shanmugalingam, Tomas Sjödin, Heli Tuominen and two anonymous referees for many useful comments given on draft versions of this book, and to Irene Zimmermann for her patience with our typesetting requests.

Particular thanks go to Outi Elina Kansanen and Juha Kinnunen who wrote draft versions for Sections 3.3–3.5. Last but not least, we are grateful to Juha Heinonen, who introduced us to analysis on metric spaces during his course in Ann Arbor in 1999.

We also encourage all future comments, in particular pointing out any mistakes, for which we apologize.

Acknowledgement. The authors have been supported by various sources during their work on this book: the *Swedish Science Research Council*, *Magnuson's fund* of the *Royal Swedish Academy of Sciences*, the *Charles Phelps Taft Research Center* at the University of Cincinnati, the *Swedish Fulbright Commission*, and *Linköpings universitet*.

They also belong to the European Science Foundation Networking Programme *Harmonic and Complex Analysis and Applications* and to the Scandinavian Research Network *Analysis and Application*.

Linköping, September 2011 Anders and Jana Björn