Contents

Preface v

1 Newtonian spaces 1
1.1 The metric space X and some notation 1
1.2 Preliminaries 3
1.3 Upper gradients and the Newtonian space $N^{1,p}$ 8
1.4 The Sobolev capacity C_p 12
1.5 p-weak upper gradients and modulus of curve families 15
1.6 Banach space and ACC_p 24
1.7 Examples 31
1.8 Notes 34

2 Minimal p-weak upper gradients 37
2.1 Fuglede's lemma 37
2.2 Minimal p-weak upper gradients 40
2.3 Calculus for p-weak upper gradients 45
2.4 The glueing lemma 48
2.5 $N^{1,p}(\Omega)$ 50
2.6 $N^{1,p}_{\mathrm{loc}}$ and D^p_{loc} 52
2.7 $N^{1,p}_0$ 55
2.8 G^p_{loc} 58
2.9 Dependence on p in g_u 59
2.10 Representation formulas for g_u 61
2.11 Notes 64

3 Doubling measures 65
3.1 Doubling 65
3.2 The maximal function 68
3.3 BMO and John–Nirenberg's lemma 70
3.4 Consequences of John–Nirenberg's lemma 75
3.5 Gehring's lemma 77
3.6 Notes 82

4 Poincaré inequalities 84
4.1 Poincaré inequalities 84
4.2 Characterizations of Poincaré inequalities 88
4.3 BiLipschitz invariance 89
4.4 (q,p)-Poincaré inequalities 91
4.5 Quasiconvexity and connectivity 99

4.6 Poincaré inequalities in quasiconvex spaces . . . 103
4.7 Inner metric . . . 106
4.8 The relation between L and λ . . . 110
4.9 Measurability . . . 113
4.10 Notes . . . 113

5 Properties of Newtonian functions 116
5.1 Density of Lipschitz functions . . . 116
5.2 Quasicontinuity of Newtonian functions . . . 123
5.3 Continuity of Newtonian functions . . . 134
5.4 Density of Lipschitz functions in $N_0^{1,p}$ . . . 138
5.5 Sobolev embeddings and inequalities . . . 141
5.6 Lebesgue points for $N^{1,p}$-functions . . . 147
5.7 Notes . . . 150

6 Capacities 154
6.1 Mazur's lemma and its consequences . . . 154
6.2 Properties of C_p in complete doubling p-Poincaré spaces . . . 157
6.3 The variational capacity cap_p . . . 161
6.4 Notes . . . 168

7 Superminimizers 170
7.1 Introduction to potential theory . . . 170
7.2 The obstacle problem . . . 172
7.3 Definition of (super)minimizers . . . 178
7.4 Convergence results for superminimizers . . . 183
7.5 Notes . . . 189

8 Interior regularity 191
8.1 Weak Harnack inequalities for subminimizers . . . 191
8.2 Weak Harnack inequalities for superminimizers . . . 197
8.3 Hölder continuity for p-harmonic functions . . . 201
8.4 The need for λ in Harnack's inequality . . . 205
8.5 Lsc-regularized superminimizers . . . 206
8.6 Lsc-regularized solutions of obstacle problems . . . 209
8.7 p-harmonic extensions . . . 212
8.8 A sharp weak Harnack inequality for superminimizers . . . 213
8.9 Notes . . . 215

9 Superharmonic functions 218
9.1 Definition of superharmonic functions . . . 218
9.2 Weak Harnack inequalities for superharmonic functions . . . 220
9.3 Lsc-regularity and the minimum principle . . . 222

9.4 Characterizations . . . 226
9.5 Convergence results for superharmonic functions . . . 229
9.6 Harnack's convergence theorem for p-harmonic functions . . . 233
9.7 Comparison of sub- and superharmonic functions . . . 234
9.8 New superharmonic functions from old . . . 235
9.9 Integrability of superharmonic functions . . . 238
9.10 Lebesgue points for superharmonic functions . . . 244
9.11 Notes . . . 246

10 The Dirichlet problem for p-harmonic functions 249
10.1 Continuous boundary values . . . 250
10.2 The Kellogg property . . . 251
10.3 Perron solutions . . . 253
10.4 Resolutivity of Newtonian functions . . . 255
10.5 Resolutivity of continuous functions . . . 261
10.6 Some consequences of resolutivity . . . 263
10.7 Resolutivity of semicontinuous functions . . . 265
10.8 The p-harmonic measure . . . 268
10.9 Poisson modification . . . 271
10.10 The resolutivity problem . . . 273
10.11 Notes . . . 275

11 Boundary regularity 277
11.1 Barrier characterization of regular points . . . 277
11.2 Boundary regularity for the obstacle problem . . . 280
11.3 Characterizations of regularity . . . 285
11.4 The Wiener criterion . . . 288
11.5 Regularity componentwise . . . 295
11.6 Fine continuity . . . 298
11.7 Notes . . . 301

12 Removable singularities 303
12.1 Removability . . . 303
12.2 Nonremovability . . . 309
12.3 Removable sets with positive capacity . . . 312
12.4 Nonunique removability . . . 314
12.5 Notes . . . 316

13 Irregular boundary points 318
13.1 Semiregular and strongly irregular points . . . 318
13.2 Characterizations of semiregular points . . . 320
13.3 Characterizations of strongly irregular points . . . 325
13.4 The sets of semiregular and of strongly irregular points . . . 327

13.5 Notes 328

14 Regular sets and applications thereof 329
14.1 Regular sets 329
14.2 Wiener solutions 331
14.3 Classically superharmonic functions 333
14.4 Notes 334

Appendices 337

A Examples 337
A.1 $N^{1,p}$ in Euclidean spaces 337
A.2 Weighted Sobolev spaces on $\mathbf{R}^n$ 340
A.3 Uniform domains and power weights 348
A.4 Glueing spaces together 349
A.5 Graphs 351
A.6 Carnot–Carathéodory spaces and Heisenberg groups 354
A.7 Further examples 357
A.8 Notes 358

B Hajłasz–Sobolev and Cheeger–Sobolev spaces 360
B.1 Hajłasz–Sobolev spaces 360
B.2 Cheeger–Sobolev spaces and differentiable structures 363

C Quasiminimizers 365

Bibliography 369

Index 389