Chapter I
Introduction

I.1 Évariste Galois 1811–1832, revolutionary mathematician

Much has been written about Évariste Galois, who died aged 20, shot in a mysterious early-morning duel in May 1832. His extraordinary mathematical intuition and his extraordinary life have, since 1846 or 1848, attracted great publicity far beyond the mathematical world. Roughly speaking, his mathematical intuition, once it was understood, changed the theory of equations from its classical form into what is now universally known as Galois Theory, together with its associated ‘abstract algebra’, including the theory of groups and fields. The essentials of his short life were captured in three words on a stamp issued in France in 1984, ‘révolutionnaire et géomètre’; they are taken from the title of a biographical work by a famous writer, André Dalmas (1909–1989) [Dalmas (1956/82)].

Évariste Galois: révolutionnaire et géomètre.

Over the years Galois’ life has been the subject of many studies, historical, biographical, fictional, dramatic, even musical: as I have said, much has been written about him. This book, however, focusses on his mathematical work and is no place to repeat it. All I offer, for contextual purposes, is a brief *cv*:

25 October 1811: Évariste Galois born in Bourg-la-Reine, about 10km south of the centre of Paris. The second of three children born to Nicolas-Gabriel Galois and his wife Adelaïde-Marie (née Demante), his sister Nathaly-Théodore was two years older, his brother Alfred nearly three years younger.

6 October 1823: entered the Collège Louis-le-Grand. His six-year stay there started well but ended badly.

August 1828: failed to gain entrance to the École Polytechnique.

April 1829: had his first article (on continued fractions) published in Gergonne's *Annales de Mathématiques*.

25 May and 1 June 1829: submitted, through Cauchy, a pair of articles containing algebraic researches to the Académie des Sciences of the Institut de France (see [Acad (1828–31), pp. 253, 257–258]). Poinsot and Cauchy were nominated as referees. The manuscripts are now lost; in [Taton(1971)] René Taton published evidence that Galois very probably withdrew them in January 1830.

2 July 1829: suicide of Évariste's father, Nicolas-Gabriel Galois.

July or August 1829: second and final failure to gain entrance to the École Polytechnique.

November 1829: entered the École Préparatoire (as the École Normale Supérieure was briefly called at that time).

Late February 1830 (probably): re-submitted his work on equations to the Académie des Sciences in competition for the Grand Prix de Mathématiques. I have not found a record in [Acad (1828–31)], but [Taton(1971), p. 137, fn 33] includes reference to a list in the archives of the Academy that contains the name of Galois. The manuscript was lost in the academy. The prize was awarded jointly to Abel (posthumously) and Jacobi for their work on elliptic functions.

April–June 1830: had three items published in Férussac's *Bulletin*.

December 1830: another item published in Gergonne's *Annales*.

4 January 1831: official confirmation of his provisional expulsion from the École Préparatoire in December 1830.

17 January 1831: submitted his 'Mémoire sur les conditions de résolubilité des équations par radicaux', now often called the *Premier Mémoire*, to the Académie des Sciences (see [Acad (1828–31), p. 566], where, however, his name is recorded as 'Le Gallois'). It was given to Lacroix and Poisson to be examined.

10 May 1831: arrested for offensive political behaviour; acquitted and released on 15 June 1831.

4 July 1831: Poisson, on behalf of Lacroix and himself, reported back negatively on the 'Mémoire sur les conditions de résolubilité des équations par radicaux': see [Acad (1828–31), pp. 660–661]; see also Note 2 to Chapter IV on p. 146 below.

14 July 1831: arrested on the Pont-neuf during a Bastille Day republican demonstration. Held in the Sainte-Pélagie prison.

23 October 1831: convicted of carrying fire-arms and wearing a banned uniform; sentenced to six months further imprisonment.

16 March 1832: released from Sainte-Pélagie prison during an outbreak of cholera in Paris and sent to live in the 'maison de santé du Sieur Faultrier', a sort of private hospital or asylum, a safe house.

Late May 1832: mysteriously engaged to duel. There is little evidence and much contradictory conjecture as to by whom and about what.

29 May 1832: wrote his *Lettre testamentaire* addressed to his friend Auguste Chevalier and revised some of his manuscripts.

30–31 May 1832: shot in an early-morning duel; died a day later in the Côchin hospital in Paris.

For more extensive accounts of his life and relations with other people see [Chevalier (1832b)], [Anon (1848)], [Dupuy (1896)], [Taton (1947)], [Taton(1971)], [Taton (1983)], [Rigatelli (1996)]. Authors do not always agree on details. For example, the first reverse at the École Polytechnique is given as June 1828 in [Toti Rigatelli (1996), p. 34], but as August 1828 by René Taton in [APMEP (1982), p. 5]. Of the second reverse [B & A (1962), p. xxviii] has it as having taken place a few days after the suicide of Galois' father, whereas [Taton(1971), p. 130] has it a few weeks later, and [Chevalier (1832b), p. 746] dates it to the end of 1829. And [B & A (1962), p. xxviii], [Taton (1947a), p. 117] have Galois entering the École Préparatoire in October 1829, whereas [Taton(1971), p. 131] has him entering in November 1829.

Évariste Galois , mort âgé de vingt et un ans , en 1832. − Ce portrait reproduit aussi exactement que possible l'expression de la figure d'Évariste Galois. Le dessin est dû à M. Alfred Galois, qui depuis seize ans a voué un véritable culte à la mémoire de son malheureux frère.

Sketch by Alfred Galois, 1848 (see p. 389).

Galois died too young to leave much evidence about his life and career, and some of what does survive is contradictory. To take a trivial example, I have often seen authors claim that Galois died aged twenty-one. As it happens, that error is an early one. Even the contemporary death certificate and autopsy report (see [Dupuy (1896), pp. 264–266]) make that mistake, as does the caption to the sketch published by his brother Alfred in [Anon (1848)]—although ‘mort âgé de vingt et un ans, en 1832’ [dead at the age of twenty-one years in 1832] could just be taken to be ambiguous and not wrong.

I would guess that the most reliable stories are those of Auguste Chevalier (1832b) and Paul Dupuy (1896). The former, though rather too heavily coloured by the sentiment proper to a close friend, and probably written mainly from memory without a great deal of checking against documents, is a contemporary account by an eye-witness. The latter, which forms the basis of most later accounts, though also coloured by sentiment, is a systematic study by a professional historian. These judgments remain, however, mere guesses on my part.

Évariste Galois, révolutionnaire et géomètre: the slogan is charmingly echoed in an ambiguity in the English title of his greatest work ‘Memoir on the conditions for solubility of equations by radicals’. But in my estimation he was far more effective as a mathematician than as a revolutionary. As a *révolutionnaire* he seems a failure; in mathematics he was a *géomètre révolutionnaire*. It is the revolutionary mathematics that we celebrate in this book.

I.2 What Galois might have read

Although this book is devoted to establishing a text rather than to interpretation, there is some background on what Galois might have read that should be helpful to readers. In particular, there are references to other writers at various points in his mathematical writings, and since these are generally exiguous it may be worth surveying briefly what we know of his mathematical reading.

He seems to have first met mathematics in his fourth year at Louis-le-Grand, when he was fifteen. According to [Anon (1848)]:

> Il dévore les livres élémentaires; parmi ces livres, il y en a un, la Géométrie de Legendre, qui est l’œuvre d’un homme d’élite, qui renferme de beaux développements sur plusieurs hautes questions de mathématiques. Galois en poursuit la lecture jusqu’à ce que le sujet soit épuisé pour lui. Les traités d’algèbre élémentaire, dus à des auteurs médiocres, ne le satisfont pas, parce qu’il n’y trouve ni le cachet ni la marche des inventeurs; il a recours à Lagrange, et c’est dans les ouvrages classiques de ce grand homme, dans la *Résolution des équations numériques*, dans la *Théorie des fonctions analytiques*, dans les *Leçons sur le calcul des fonctions*, qu’il fait son éducation algébrique.
>
> [He devours the elementary books; among these books there is one, Legendre’s Geometry, the work of a distinguished man, which contains

> beautiful developments on several deep questions in (higher) mathematics. Galois takes his reading of it to the point where the subject is exhausted for him. The elementary treatises on algebra, by mediocre authors, do not satisfy him because he finds in them neither the authority nor the steps of the discoverers; he turns to Lagrange and it is in the classic works of this great man, in the *Résolution des équations numériques*, in the *Théorie des fonctions analytiques*, in the *Leçons sur le calcul des fonctions*, that he acquires his algebraic education.]

In his biographical study Dupuy made a similar report [Dupuy (1896), p. 206], though most of it relies on this passage. The works referred to here are, presumably, one or other of the editions of the *Éléments de géométrie* [Legendre (1823/1799)], the *Résolution des équations numériques* [Lagrange (1798)], the *Théorie des fonctions analytiques* [Lagrange (1796/1813)], the *Leçons sur le calcul des fonctions* [Lagrange (1803)]. That Galois had read and understood this last book is confirmed by a reference to it in one of the manuscripts (**f. 189 b**, [B & A (1962), p. 413]), where he criticises and seeks to correct it.

It seems likely, as suggested in [Ehrhardt (2010a), p. 95], that Galois acquired some of his knowledge of the theory of equations from standard textbooks of the time such as the two books by Lacroix, *Élémens d'algèbre* and *Complément des Élémens d'algèbre* [Lacroix (1799)], [Lacroix (1800/1835)]; another such (see [Dhombres (1984)], [Dhombres (1985)]) was Bezout's widely read *Cours de mathématiques* [Bezout (1820/1770)]. It is not unreasonable to believe that he would also have read, or at least dipped into, such classics as the *Élémens d'Algèbre* of Clairaut [Clairaut (1801/1740)] and of Euler [Euler (1807)]. His deeper knowledge will probably have come, though, from the second or third editions of such monographs as [Lagrange (1798)] cited above on equations or Legendre's *Théorie des Nombres* [Legendre (1798/1808)] (with its supplement (1816) on numerical solution of equations), from Gauss's *Disquisitiones arithmeticae* (1801), or from Cauchy's *Cours d'analyse* (1821). Presumably he also read the issues of the main journals as they came out: Gergonne's *Annales* and Férussac's *Bulletin* in which he published; the publications of the Academy of Sciences; Crelle's *Journal*; Cauchy's *Exercices*; possibly also the publications of some of the foreign academies.

It would be interesting to know whether Galois had read the great and influential memoir 'Réflexions sur la résolution algébrique des équations' [Lagrange (1770/71)]. That is possible. But in his reaction to the note in which Poisson refers to it (see Ch. IV, Note 8, p. 153) Galois does not acknowledge the connection with his own work. I estimate it to be more likely that he acquired his deeper understanding of the algebraic solution of equations (that is to say, his understanding beyond what is to be found in texts such as those of Lacroix and Euler) from the précis of that memoir which is the content of Note XIII appended to the second and third editions of [Lagrange (1798)].

We have little detailed evidence, but it seems safe to conjecture that on elliptic functions he would surely have read Legendre's *Exercices de Calcul Intégral*

[Legendre (1811)] or his *Traité des fonctions elliptiques* [Legendre (1825–28)], and also the work of Jacobi and Abel. In respect of Jacobi there is a morsel of evidence to confirm this belief in a draft of a letter from Alfred Galois (see p. 388), though the evidential value of the passing reference there seems slight—would Alfred really have understood much of what his older brother studied? Is this claim that Évariste studied Jacobi's work deeply any more than a courtesy?

Unfortunately, because Galois had not been trained to cite his sources and, as shown in his writings, did not have an innate instinct to do so, we cannot know for certain what he had read. Nevertheless, there are some indications in what he left us. We can be sure, for example, that he had read the *Disquisitiones arithmeticae* [Gauss (1801)], not because he cites the work explicitly but because for the majority of his references to Gauss this is the only work that is relevant (see pp. 50, 62, 86, 108, 130, and see also [Neumann (2006)]).

I do not find it easy to estimate with any degree of precision what of Cauchy's writings Galois had read. We can be sure that he had read Cauchy's first article on substitutions [Cauchy (1815a)] both because it is cited explicitly at one point (p. 128) and because he discussed questions from it (see Dossier 15, p. 284). He used language from the article on substitutions and determinants [Cauchy (1815b)], but he might have got that from the *Cours d'Analyse* [Cauchy (1821), Ch. III, p. 73; Note IV, p. 521] (see Note 10 to Chapter VI, p. 296 below). But of course, having read [Cauchy (1815a)] he might naturally have let his eye stray to [Cauchy (1815b)].

There is a very interesting question as to what Galois had read of Abel's work. This will be treated briefly in Note 6 to Dossier 10 (p. 242 below).

I.3 The manuscripts

Soon after his death the manuscripts that Galois left on his desk came into the hands of his friend Auguste Chevalier, who made copies of a number of them. Some time in the summer of 1843 Chevalier gave them to Joseph Liouville (1809–1882) (see [Liouville (1843)]), who left them (included in his library of books and papers) to his son-in-law Célestin de Blignières (1823–1905). They were sorted by Mme de Blignières, daughter of Liouville and widow of de Blignières, and given to the Académie des Sciences in 1905 or 1906 (see [Tannery (1906), p. 226]). They are organised into 25 dossiers bound into one volume catalogued as Ms 2108 in the library of the Institut de France. High quality facsimiles of some of the pages have been published in, for example, [B & A (1962)] and [APMEP (1982)].

Almost all the manuscripts were written in ink, though a very few of the pages in Dossier 24 contain material in pencil. They are now very fragile and, unless one has special privileges, what one used until recently was a microfilm copy or a two-volume printout made from it. Through the efforts of Mme Sylvie Biet, Mme Annie Chassagne, and others in the library, however, very good digital images made by

Mr F. Xavier Labrador have, since 15 June 2011, been mounted on the web. They are accessible from the page listed at [Galois (2011)] (see p. 392), which makes the whole of Ms 2108 publicly available for the first time.

Each dossier has a cover sheet. The organisation into dossiers was probably done by Mme de Blignières—if such a conjecture does not read too much into Tannery's words [Tannery (1906), p. 226] 'M$^{\text{me}}$ de Blignières s'occupe pieusement de classer les innombrables papiers de son mari et son illustre père' [Mme de Blignières is piously busy classifying the innumerable papers of her husband and her illustrious father]. The cover sheets, however, seem to have been annotated by the librarian or perhaps by Jules Tannery himself in or after 1908. Each has a brief description of the contents forming a sort of title, together with page-references to the 1897 Picard edition of the main works or to the 1906/07 paper by Tannery and its reissue in 1908 as a book.

The cover sheet of Dossier 1 (the First Memoir) exemplifies this, but also carries an extra explanatory note. It is inscribed as follows:

Mémoire sur les
Conditions de résolubilité des équations par radicaux.
(Oeuvres, p. 33)
(Texte autographe du Mémoire présenté à l'Académie)

Below that header and on the left of the page, as if intended as a marginal comment, is the following 9-line note:

> Les renvois aux "Oeuvres"
> se rapportent à l'Edition
> des "Oeuvres Mathématiques
> d'Evariste Galois" publiée
> par la Société Mathématique
> Gauthier-Villars 1907.
> Les renvois marqués (M.) se
> rapportent aux "Manuscrits
> de Evariste Galois" Gauthier-Villars 1908.
> J. T.

> [The references to the "Oeuvres" refer to the edition of the mathematical works of Evariste Galois published by the Mathematical Society, Gauthier-Villars 1907. The references marked (M.) refer to the "Manuscrits de Evariste Galois" Gauthier-Villars 1908. J. T.]

The reading of the second initial here is uncertain: my reading of it as T reflects a conjecture that the cover-sheet was provided (or at least, annotated) by Jules Tannery. Whether or not that is correct, whoever it was, the writer seems to have been sufficiently familiar with the editions of Galois' works and manuscripts to have become a little blasé—to the point where he or she did not feel the need to check, and has quoted the references somewhat imprecisely. The relevant edition of the "Œuvres Mathématiques d'Évariste Galois" is [Picard (1897)], and the reference to the manuscripts is to [Tannery (1908)]. But perhaps it was not Jules Tannery at all—perhaps the explanation was written by someone only partially familiar with the editions of Galois' works, perhaps someone such as the librarian of the Institut de France who had early charge of the Galois manuscript material. In any event, it is clear that this marginal note refers to annotations on the cover sheets of the dossiers into which the manuscripts are organised.

I.4 Publication history of Galois' mathematical writings

Five of Galois' mathematical papers were published in his lifetime. The most important works, however, are posthumous. On 29 May 1832, the eve of the fatal duel, Galois wrote his famous *Lettre testamentaire* to his friend Auguste Chevalier, a letter that summarised the mathematics he was storing in his mind and also, in effect, asked (or commanded) Chevalier to act as his literary executor. This letter was duly published, as Galois had requested, in September 1832. Liouville published his highly influential edition [Liouville (1846)] of the main works as an article in his *Journal de Mathématiques pures et appliquées*, the journal that he had founded in 1835 as a successor to Gergonne's *Annales de Mathématiques pures et appliquées* and which he edited for many years. The most influential items are

- 'Sur la théorie des nombres' published in 1830 (see [Galois (1830c)]), in which Galois produced his theory of what used to be called 'Galois imaginaries', most of what later became the theory of finite fields;
- 'Mémoire sur les conditions de résolubilité des équations par radicaux', also known as the *Premier Mémoire*, the paper that was rejected by the Académie des Sciences on 4 July 1831 and returned to its author, but which gave us what we now call Galois Theory;
- 'Des équations primitives qui sont solubles par radicaux' also known as the *Second Mémoire*;
- the letter of 29 May 1832 to Auguste Chevalier known as the *Lettre testamentaire*.

The *Œuvres* as published by Liouville were reprinted and issued in book form by Picard in 1897 for the Société Mathématique de France.

As has already been mentioned, the *Lettre testamentaire* was published (at Galois' express request) by Chevalier in September 1832 in the *Revue Encyclopédique*. It has been reprinted in all editions of Galois' works since 1846. Almost certainly (see Ch. III, Note 2, p. 101) that first publication was done from a copy made by Chevalier which was kept by the printer: no copy in Chevalier's hand is still extant, whereas the original still exists as the contents of Dossier 2 of the Galois manuscripts.

Jules Tannery published a two-part paper in the *Bulletin des Sciences Mathématiques*, 1906 and 1907. It is devoted to a comparison of the 1897 edition with the manuscripts and to the physical description and publication of some (the majority, in fact) of the lesser manuscripts.

Then in 1962 Robert Bourgne and Jean-Pierre Azra produced their great critical edition [B & A (1962)] of the *Écrits et Mémoires Mathématiques d'Évariste Galois*. Everything is collected and re-published in one volume. The manuscripts are described; crossed-out material is deciphered; Galois' insertions and afterthoughts are recorded. A second edition was published in 1976. The changes were minimal, however: simply the addition of eight unpaginated leaves (two of them blank) inserted between pages *xvi* and *xvii*, containing an *errata* list and two tables of editorial information on pagination. Minimal they may be, but they are useful and respond well to points made by Taton in his review [Taton (1964)].

The various editions discussed here are listed at the beginning of the bibliography on p. 391.

In 1889 there was a translation of most of the Liouville edition [Liouville (1846)] into German [Maser (1889)]. Much more recently there have been Italian translations of the main works [Toti Rigatelli (2000)] and of the schoolwork [De Nuccio (2003)] (I have not seen these books and owe the references to Professor Massimo Galuzzi). Translations of a few items into English have appeared from time to time. For example [Smith (1929)] contains a translation of the *Lettre testamentaire* and [Edwards (1984)] contains a translation of the *Premier Mémoire*. There are also many snippets in various source-books, and the first half of the *Second Mémoire* is translated in [Neumann (2006)]. Taken altogether, however, only about one-third of the Galois *œuvre* has been available in English up until now; moreover that one-third is to be found in a variety of disparate sources.

That is the reason for the present new edition. An English edition is, however, of little use without the original alongside for direct comparison. I had originally planned simply to use [B & A (1962)] as the French version, but it soon became clear that that would not do. For one thing, Bourgne & Azra used the facing page, which is needed here for the translation, for editorial purposes. Without that editorial material the main text of Galois' writings becomes less valuable and it therefore had to be incorporated into the French transcription. For another, comparison with the manuscripts to clear up a few points in [B & A (1962)] that had puzzled me for some forty years led me to recognise that it was somewhat less perfect than I had always believed. Therefore my French edition is not simply a re-issue of the older one; it is a new transcription. It has been as carefully checked against both the old editions and the original manuscripts as I am capable of. This is very detailed work and we must accept, I am sorry to say, a high probability that I have made mistakes. I much hope, however, that there will turn out to be few of them.

I.5 The reception of Galois' ideas

A brief account of the reception of Galois' ideas should provide some more context. Extensive treatments are offered in [Kiernan (1971)] and [Ehrhardt (2007)]. Here is a mere summary of the highlights.

After the publication of Liouville's edition [Liouville (1846)] Galois' ideas, though not easily understood in those days, spread steadily through France, Italy and Germany. Liouville had already lectured on them, perhaps in the winter of 1843–44 (see [Bertrand (1899), p. 398], [Lützen (1990), p. 131]), and Serret had attended. The first edition [Serret (1849)] of his *Cours d'algèbre supérieure* contains (p. 4) a summary and high praise for the work of Galois (spelled Gallois); on pp. 149–152 there is a careful account of Galois' Lemme III from the *Premier Mémoire*; but in the footnote to p. 344 Serret makes clear that he had not yet fully understood the ideas of Galois in their entirety. When he came to publish the much enlarged third edition [Serret (1866)] however, he was able to include a pretty full account of the material in the *Premier Mémoire* (Vol. 2, pp. 413–420; 607–647). I would guess that his understanding had developed through conversations with his pupil Camille Jordan, who had come to terms with Galois' ideas in the early 1860s. See [Lützen (1990), pp. 129–132, 196–197] for an excellent account of Serret and his relationships with Liouville and with Galois Theory. Jordan's writings, [Jordan (1861), Supplément], [Jordan (1865)], [Jordan (1867)], [Jordan (1869)], and especially the great *Traité des substitutions et des équations algébriques* (1870), show that he had understood Galois' ideas to the level where he could develop them, as Serret had not. As I wrote in [Neumann (2006), p. 414]:

> The *Traité* is described by its author as being nothing but a commentary on the works of Galois "[...] les Œuvres de Galois, dont tout ceci n'est qu'un Commentaire" ['... the Works of Galois, of which all this is no more than a Commentary'] (see [Jordan (1870), p. viii]). Some commentary! It is 667 quarto pages.

Meanwhile, Betti had published several papers in Italy seeking to elucidate Galois' work, of which the main ones are [Betti (1851)], [Betti (1852)]. These are not entirely successful, and their shortcomings are analysed in [Mammone (1989)] (but see my review of this paper in *Mathematical Reviews* 1991, Review 91j:01026). In Germany Kronecker wrote a little about the theory of equations, focussing more on Abel's work than on that of Galois, though he did discuss the Galois theory of irreducible equations of prime degree; Dedekind also published rather little, but he lectured on Galois Theory (see [Scharlau (1982)]); Netto's books [Netto (1882)], [Netto (1892)], heavily based on Jordan's *Traité* (in spite of some ill-feeling between Netto and Jordan in the early 1870s after the Franco–Prussian war), brought Galois Theory to a wider public both in Germany and in America; and towards the end of the century Weber's article [Weber (1893)] and his famous and very influential textbook [Weber (1895)] were published.

In the 1850s and 1870s Cayley famously tried to develop an abstract theory of groups (citing Galois for the word *groupe* in a footnote), but he did not seem to understand Galois' ideas to any depth, and Galois Theory did not take hold in Britain until the 20th century. The famous textbook [Burnside & Panton (1881)], for example, went through many editions from 1881 until the mid-1920s (and was reprinted by Dover Publications, New York, in 1960), but, beyond brief mention of Galois in the context of its treatment of substitutions, it expounds very few of his ideas. The third edition (1892) has his Lemma III from the First Memoir. Vol. 2 of the fourth edition (1901) introduces groups and a little group theory in Cauchy's style (this looks to be heavily based on Serret's treatment in his *Cours d'algèbre supérieure*). It has a definition of the Galois group, but does not explain Galois Theory even although it deals with insolubility of the general equation of the fifth degree in the manner of Wantzel's version of Abel's proof. Near the end of the 19th century Oskar Bolza and James Pierpont gave series of lectures that brought Galois Theory to America (see [Bolza (1890)], [Pierpont (1899)], [Pierpont (1900)]).

The above paragraphs treat the development of Galois Theory, but the theory of groups was developing not only as a part of Galois Theory but also as a subject in its own right. It came from two more-or-less independent sources, namely the publication of the *Œuvres* of Galois in 1846 and the publication by Cauchy of about 25 notes in the *Comptes rendus hebdomadaires de l'Académie des Sciences*, of which the first four are [Cauchy (1845a)], and a long article [Cauchy (1845b)] that overlaps considerably with the *CR* notes. His approach was different from that of Galois, as was his language. What was a *groupe de substitutions* in the writings of Galois was a *système de substitutions conjuguées* in those of Cauchy. The two approaches were complementary. They came together in the work of Camille Jordan who, in his thesis [Jordan (1860)], [Jordan (1861)] used the language of Cauchy to treat the Academy problem that had been announced for the Grand Prix de Mathématiques for 1860 (a problem that had come out of Cauchy's work), but who quickly came to understand and develop the ideas of Galois (see, for example, [Jordan (1865)], [Jordan (1867)], [Jordan (1869)], [Jordan (1870)], [Neumann (2006)]). There have been many studies of the development of the theory of groups in the 19th century and the reader is referred to [Wussing (1969)], [Neumann (1999)] and references cited in those works for fuller information.

I.6 Scope of this edition

Included here is everything published by Liouville and by Tannery, and a little more. Exigencies of time and space prevented me from including everything published by Bourgne & Azra. Theirs remains the only complete edition of the writings of Galois.

The material is organised as follows. First come the five mathematical articles published while Galois was alive. From this period I have excluded only the letter 'Sur l'Enseignement des Sciences: des Professeurs, des Ouvrages, des Examinateurs' published in the *Gazette des Écoles* on 2 January 1831. Although it contains Galois reflections on the study of mathematics in the colleges of Paris it is not, I find, a particularly edifying piece, and contributes little to our understanding of the mathematician in Galois.

After the published articles comes the Testamentary Letter written on 29 May 1832, the eve of the duel. I place it there for two reasons: first because it was the next to be published (in September 1832); secondly because it includes an admirable synopsis of the substance of Galois' discoveries.

Then come the manuscripts essentially in the order in which they appear in the collection in the Institut de France: the great First Memoir, which, when it was first published by Liouville in 1846 quickly led to the development of Galois Theory and group theory; then the Second Memoir, also first published by Liouville in 1846; finally the minor manuscripts, most of which were first published by Tannery in 1906/07.

Some of the minor manuscripts, those in Dossiers 9–14, contain little mathematics. They could be described as philosophical-polemical. Nevertheless, they seem to have been intended by Galois as part of his mathematical work. He had intended to write some expositions of algebra; he apparently dreamed of publishing his First and Second Memoirs, or something like them, as a small book; and these seem to have been conceived as introductory material. This, at least, is how Auguste Chevalier seems to have interpreted them.

Each item is preceded by an introductory page giving information about previous editions, physical descriptions of the manuscripts, jottings, and other such matters. Most items are followed by notes intended to supply context. I have tried to restrain myself from exegesis. Thus the commentary is focussed on content, not meaning; on syntax, not semantics; on relationships with previous editions. I much hope to find time in the future to write articles dealing with various parts of the mathematics produced by Galois, articles similar to [Neumann (2006)], which deals with just the first few pages of the *Second Mémoire*. But for this book I have tried to suppress my mathematical instincts.

Missing from this edition are the many scraps containing scribbles and partial calculations. These are to be found in [B & A (1962), pp. 189–361] and since no translation is needed (or indeed possible) there is no point in copying them here. Also missing are the items of schoolwork published in [B & A (1962), pp. 403–458]. These would have merited inclusion had time and space permitted.

I.7 Editorial ambition and policy

Leaving aside Gergonne and Sturm (for Férrusac's *Bulletin*), the principle editors of the Galois manuscripts were Auguste Chevalier, Joseph Liouville, Jules Tannery, and Robert Bourgne & Jean-Pierre Azra.

I.7.1 Auguste Chevalier

Auguste Chevalier was a close friend of Galois, and was, in effect, appointed by him as his literary executor (see the end of the Testamentary Letter). The obituary [Chevalier (1832b)] he published in November 1832 begins

> Il y a trois ans bientôt que j'ai connu Galois; notre liaison commença à l'Ecole Normale, où il entra un an après moi.
>
> [I have known Galois for nearly three years; our relationship began at the Ecole Normale, which he entered a year after I did.]

He was the first editor of the Galois manuscripts. He had the Testamentary Letter published in the *Revue Encyclopédique* [Lettre (1832)], as Galois had commanded, and he made copies of the *Premier Mémoire*, the *Second Mémoire*, the *Discours Préliminaire* (Dossier 9), the *Préface* (Dossier 11), and the *Discussions sur les Progrès de l'Analyse pure* (Dossier 12). His manuscripts are bound in with the Galois manuscripts in the library of the Institut de France. Apart from somewhat erratic use of capital letters, Chevalier's copies are remarkably faithful and accurate. The few places where he made small corrections are indicated in my marginal notes.

I.7.2 Joseph Liouville

The Galois manuscripts came to Liouville from Chevalier some time in the summer of 1843 (see [Liouville (1843)]). He planned to publish at least the *Premier Mémoire* that same year, as is proved by the existence at the end of Dossier 3 (of the Galois manuscripts) of corrected proof sheets carrying the reference 'Tome VIII, DÉCEMBRE 1843'. The material appeared three years later as an item in the great edition [Liouville (1846)]. The delay may have been due to difficulties with the material. In his 1843 announcement to the Académie des Sciences, Liouville said:

> Le Mémoire de Galois est rédigé peut-être d'une manière un peu trop concise. Je me propose de le compléter par un commentaire qui ne laissera, je crois, aucun doute sur la réalité de la belle découverte de notre ingénieux et infortuné compatriote.
>
> [Galois' memoir is written in a style that is perhaps a little too concise. I propose to complete it with a commentary which will leave no doubt, I believe, as to the correctness of the beautiful discovery of our ingenious and unfortunate compatriot.]

No mathematical commentary accompanied the 1846 publication of the *Œuvres*. The last item in Dossier 3, however, is a manuscript by Liouville supplying a proof of Proposition II (see Ch. IV, Note 16, p. 159), to be inserted into the aborted 1843 printed version of the *Premier Mémoire* at the point where Galois had left the marginal note 'Il y a quelque chose à completer dans cette démonstration. Je n'ai pas le tems.'

[There is something to be competed in this proof. I do not have the time.] Moreover, Dossier 27 contains some 25 pages of manuscript notes by Liouville working out Galois' ideas. A fine account of these notes is to be found in [Lützen (1990), pp. 567–577] .

Liouville's job as editor of the *Journal de Mathématiques pures et appliquées* was to make mathematics available to a wide mathematical public. Thus he very properly corrected errors of grammar and spelling, and even slips in the mathematics. He also had lemmas, theorems and formulae displayed as is usual in mathematical publication. He would have done the same for any author. He did his job well. See [Lützen (1990), pp. 579–580] for an analysis of his impact. The fact is that Galois' ideas, though published in synoptic form in the *Lettre testamentaire* in September 1832, were essentially lost to the mathematical public after his death. It was as if they had been buried with Galois. Liouville not only disinterred them, he gave them the full life that they deserved.

Picard is not listed above among the principal editors because his 1897 edition followed that of Liouville pretty closely. He too was concerned to make Galois' ideas easily available, for the first time in book format, to mathematical colleagues. Since Liouville's (and Picard's) corrections are mostly routine very few have been noted in my marginal annotations.

I.7.3 Jules Tannery

Tannery was an editor of a different kind, concerned quite as much with the historical importance of the Galois material as with its mathematical content—after all, by the end of the 19th century the mathematics had been fully developed and taken a long way beyond what Galois himself had created. In [Tannery (1906), pp. 227, 229] he wrote:

> L'importance de l'œuvre de Galois sera mon excuse pour la minutie de certains détails, où j'ai cru devoir entrer, et qui va jusqu'au relevé de fautes d'impression, dont le lecteur attentif ne peut manquer de s'apercevoir. Je ne me dissimule pas ce que cette minutie, en elle-même, a de puéril. [...]
>
> J'ai collationné le manuscrit avec le texte imprimé: [...]

> [The importance of Galois' work will be my excuse for the extreme care over certain details that I have believed I should enter into, extending as far as the listing of printing errors which the attentive reader could not fail to notice. I am well aware that, in itself, this extreme care has a trifling element. [...]
>
> I have collated the manuscript with the printed text.]

In [Tannery (1907), p. 279] he added

> Quant aux fragments qui suivent, j'ai cru devoir les reproduire tels quels, avec une exactitude minutieuse, en conservant l'orthographe, la ponctuation ou l'absence de ponctuation, sans les quelques corrections qui se présentent naturellement à l'esprit. Cette minutie m'était imposée pour les quelques passages où la pensée de Galois n'était pas claire pour moi; sur cette pensée, les fragments informes que je publie jetteront peut-être quelque lueur. Je me suis efforcé de donner au lecteur une photographie sans retouche.
>
> [As for the fragments which follow, I have believed I should reproduce them as they are, with minute exactitude, preserving the spelling, the punctuation or the absence of punctuation, without the few corrections that naturally occur to one's mind. This great care was imposed upon me for the few passages where the thinking of Galois was not clear to me; the imperfect fragments which I am publishing will perhaps throw some light on this thinking. I have made great efforts to give the reader an un-retouched photograph.]

I.7.4 Robert Bourgne & Jean-Pierre Azra

Bourgne & Azra were editors of the same kind as Tannery, and guided by the same principles. Following a similar line to his, indeed, echoing some of it, Robert Bourgne [B & A (1962), p. xv] wrote:

> Ce livre rassemble tout ce que nous avons conservé d'Evariste Galois, mémoires, articles, recherches, brouillons, lettres. [...]
>
> On l'a fait pour qu'il livre au mathématicien un texte exact et complet, pour qu'il offre à l'historien de quoi préciser un grand moment de la pensée mathématique. Il ne s'agissait que de lire scrupuleusement les manuscrits et, s'ils manquent, de revenir à la publication originale. Point de retouche. Nous livrons la copie exacte. Nous avons respecté la ponctuation de Galois et maintenu les distractions du manuscrit ou les fautes du texte original. Si la correction s'impose, on la signale en note.
>
> Nous avons déchiffré toutes les ratures. [...]
>
> [This book gathers together all that is preserved for us of Evariste Galois. [...]
>
> It has been made [written] in order to deliver to the mathematician a correct and complete text, in order to offer to the historian something to define a great moment in mathematical thought. There was nothing to be done but to read the manuscripts scrupulously and, where they are missing, to return to the original publication. Absolutely no retouching. We deliver an exact copy. We have respected Galois' punctuation and

> retained the slips in the manuscript or the original text. Where correction is required it is indicated in a note.
> We have deciphered all the crossings-out.]

I have found that most of Galois' errors of spelling and punctuation have been transcribed faithfully in Tannery's edition of the minor manuscripts, but a great many of them have been silently corrected in [B & A (1962)]. There are so many discrepancies between manuscript and print (well over 500 of them), and most of them are so trivial, that I do not have the space or the stomach to record other than a few of the more surprising and significant ones.

Although [B & A (1962)] is clearly the joint work of both the editors whose names appear on the title page, so that it would normally be correct to write such phrases as 'Bourgne & Azra noted that', it is clear from various passages in their book that they themselves distinguished their contributions. The *Avertissement* [Preface] to the book, for example, is signed by Robert Bourgne alone, whereas a second *Avertissement* attached to Chapter III of the Fourth Part ('Derniers vestiges: Brouillons et calculs inédits' [Last remains: scraps and unpublished calculations]), [B & A (1962), pp. 189, 191], is signed by J. P. Azra alone. I shall therefore sometimes take the liberty of attaching one name or the other to various passages. Since the present book has little overlap with the part of [B & A (1962)] for which Azra was responsible, most of those references will be to Robert Bourgne.

I.7.5 The present work

In the present edition, following the precedents set by Tannery and Bourgne & Azra, the French transcription is intended as a 'warts and all' version of the manuscripts (I return to what this should mean below). My method was this. A few pages were transcribed directly from the manuscripts, but the majority were taken first from a printed source. Some of the pages came from [Liouville (1846)] by download from the *Gallica* digital edition, some from [Tannery (1906)], [Tannery (1907)] by download from *Gallica* (the digital library of the Bibliothèque Nationale de France), and a very little from [B & A (1962)] by transcription. All were then checked against the original publications (for the material published in Galois' lifetime) or the manuscripts as appropriate.

My marginal annotations refer mainly to discrepancies between the manuscripts and the various editions. Although some of these are significant in relation to an ambition to establish a text, none of them is important mathematically, and small print is entirely appropriate for them.

In the notes (and sometimes in the text) the following code is used:

- *ms*: the manuscripts written by Galois himself;
- C*ms*: Chevalier's manuscript copies;
- C1832: Chevalier's edition [Lettre (1832)] of the *Lettre testamentaire*;
- L1846: Liouville's edition [Liouville (1846)] of the main works;

P1897: Picard's reissue of Liouville's edition in the reprint [P & T (2001)];
T1906/7: Tannery's paper of 1906–1907 in the reprint [P & T (2001)];
BA1962: The Bourgne & Azra edition [B & A (1962)] of 1962/76.

Since P1897 is a close copy of L1846 (except in some typographical matters) it is, in fact, rarely referenced.

I.8 Translation and interpretation

The English differs from the French in some significant ways. In particular, in trying to establish an English edition I have followed the usual conventions for published mathematics. Thus, for example, italics are used where the manuscript has underlined words; italics are used (in most places—though not in Dossier 15) for the statements of propositions, theorems, lemmas, and the like; punctuation is corrected and modernised. Most, but not quite all, of the crossed-out material is incorporated into the translation. This was done mainly in the hope that some readers might find it useful, partly to help the English and French run properly in tandem on opposite pages. I can only hope that it does not make difficulties for those readers who want just the main text.

Generally I have tried to produce a rather literal translation, so as to give a fair idea in English of what Galois actually wrote. The reader will therefore find some artificial phrases and some sentences which could be re-cast into more agreeable form. Reading Galois is not difficult. He writes a legible and (mostly) pleasant hand. Understanding what he is saying is not difficult either (except where, in the polemical passages, his writing becomes unsympathetic and idiomatic). Translating, however, is harder. Many words and phrases, both in French and in English, have changed their usage and/or their meaning over these last 200 years. I can only hope that, having the original side by side with the translation, the reader will be able to make the comparisons which should lead to a good understanding of what Galois was trying to explain.

One example should give an idea of what I mean by the difference between understanding and translation. Near the beginning of the *Lettre testamentaire* (see p. 84) there is the sentence

> Le premier [mémoire] est écrit, et malgré ce qu'en a dit Poisson, je le maintiens avec les corrections que j'y ai faites.

The mention of Poisson refers to the report he made to the Académie des Sciences on 4 July 1831 (see Ch. IV, Note 2, p. 146). Although it is quite clear what Galois meant, finding a good English equivalent for his word *maintiens* presents difficulties. Dictionaries variously give 'maintain', 'keep', 'sustain', 'endorse', 'uphold', 'support', for the transitive French verb *maintenir*. But in this context none of these sounds quite right to my ear. My own translation is

> The first is written, and in spite of what Poisson has said about it I stand by it with the corrections that I have made in it.

In [Weisner (1929), p. 278] it is rendered as

> The first is written, and, despite what Poisson has said of it, I am keeping it, with the corrections I have made.

I find this an agreeable translation except that 'I am keeping it' does not quite fit the context. [Fauvel & Gray (1987), p. 503, § 15D1]—quoted in [Wardhaugh (2010), p. 21], where, however, it is mis-attributed to Weisner in [Smith (1929)]—offers

> The first is written, and despite what Poisson has said about it, I hold it aloft with the corrections that I have made.

Although 'hold it aloft' could be a version of 'uphold', this does not, I feel, present quite the picture that Galois would have had in mind. Perhaps the fact that Chevalier mistranscribed it as *soutiens* is an indication that the word *maintiens* is not particularly natural in this context.

Technical terms are a particular problem because translation involves judgments of meaning, so that translation and interpretation have to go together. Here are some notes on some of the more important words.

I.8.1 The words *analyste* and *géomètre*

The two words *analyste* and *géomètre* are fine illustrations of the problem with meanings. Their natural translations into modern English are of course 'analyst' and 'geometer' respectively (though a brass plate on an office door announcing *M. Pierre Lemesurier, Géomètre* says that Mr Pierre Lemesurier is a surveyor). In 1830, however, the two words both had two main meanings: on the one hand they were both used to mean 'pure mathematician' generically, though the former had overtones of 'algebraist'; on the other hand they indicated practitioners of the main subdivisions of pure mathematics.

They derive of course from the nouns *analyse* and *géométrie*. These were parts of *mathématiques* or *sciences mathématiques*, a broad area that included both pure and applied mathematics and more besides. Thus, for example, Vol. 21 (1830/31) of Gergonne's *Annales de mathématiques pures et appliquées* organised the material under 12 headings:

- *Analyse Algébrique;*
- *Analyse Appliquée;*
- *Analyse Élémentaire;*
- *Analyse Indéterminée;*
- *Analyse Transcendante;*
- *Arithmétique;*

- *Arithmétique Sociale;*
- *Géométrie Analytique;*
- *Géométrie des Courbes;*
- *Géométrie Élémentaire;*
- *Géométrie Transcendante;*
- *Philosophie Mathématique.*

Other volumes have similar lists, and whereas topics like *arithmétique, arithmétique sociale, astronomie, dynamique, hydrodynamique, hydrostatique, météorologie, philosophie mathématique, optique* come and go, various branches of *analyse* and *géométrie* always appear. Roughly speaking *analyse* covered algebra, number theory and calculus while *géométrie* covered spatial matters.

If the above paragraph gives an impression that *analyste* and *géomètre* were more or less synonymous then it is wrong. For one thing, as was indicated above, the former carries the suggestion of 'algebraist', if in a broad sense. For another, although pure mathematics was pretty much covered (with some overlap) by *analyse* and *géométrie*, the latter was the broader term. From 1820 to 1835, of the eleven sections of the Académie des Sciences, there was only one—as one sees when one reads the published minutes, the *Procès-Verbaux de l'Académie des Sciences de l'Institut de France*—that naturally covered pure mathematics and that was the *section de géométrie*. There were members of the *sections d'astronomie, de mécanique, de physique générale*, and perhaps even of other sections, who wrote articles that we might naturally classify as pure mathematics. These would then, however, be thought of as contributions to *géométrie*. There was no *section de mathématiques*.

During the first half of the 19$^{\text{th}}$ century the word *mathématiques* occurred quite rarely—hardly more than in the title of Gergonne's *Annales*, the title of Liouville's *Journal*, and in the context of the Academy prizes (*prix de mathématiques, grand prix de mathématiques*). Thus in the language of the Academy the word *géomètre* was used in the same way as we might use 'pure mathematician'. Abel, for example, contributed much more to *analyse* than to *géométrie*, but could nevertheless be referred to by Galois as a *géomètre* (see, for example, Section VI.5), meaning simply 'mathematician'. And the 1984 French postage stamp portraying Galois and describing him as '*révolutionnaire et géomètre*' is saying that he was a revolutionary and a mathematician—he was, after all, very much an algebraist and analyst, rather than a geometer in the familiar modern senses of these words.

I.8.2 The phrases *équation algébrique* and *équation numérique*

The phrases *équation algébrique* and *équation numérique* also require some thought. Their natural literal translations are 'algebraic equation' and 'numerical equation', meaning equations with literal or numerical coefficients, respectively. In almost all contexts in 18$^{\text{th}}$- and 19$^{\text{th}}$-century writings, however, the adjective does not in fact qualify the noun equation. It refers instead to what the writer has in mind as a strategy for solution. The former refers to the search for a formula, the latter to iterative numerical methods. Thus Lagrange's great works [Lagrange (1767)], [Lagrange (1798)], with titles involving *la résolution des équations numériques* have the adjective numerical qualifying the plural noun equations, and yet their subject is numerical methods for finding accurate approximations to the roots of polynomial equations.

A paragraph in Poinsot's preface to the 1826 edition of [Lagrange (1798)] (originally a review of the 1808 edition published in the *Magasin Encyclopédique* in 1808) explains well:

> D'abord si l'on jette un coup d'oeil général sur l'Algèbre, on voit que cette science, abstraction faite des opérations ordinaires (au nombre desquelles on peut compter l'élimination), se partage naturellement en trois articles principaux. 1$^{\text{o}}$. La théorie générale des équations, c'est-à-dire l'ensemble des propriétés qui leur sont communes à toutes. 2$^{\text{o}}$. Leur résolution générale, qui consiste à trouver une expression composée des coefficiens de la proposée, et qui, mise au lieu de l'inconnue, satisfasse identiquement à cette équation, en sorte que tout s'y détruise par la seule opposition des signes. 3$^{\text{o}}$. La résolution des équations numériques, où il s'agit de trouver des valeurs particulières qui satisfassent d'une manière aussi approchée qu'on le voudra, à une équation dont tous les coefficiens sont actuellement connus et donnés en nombres.
>
> [If one casts a general glance over algebra, one sees first that, setting aside ordinary operations (numbered among which one can include elimination), this science is divided naturally into three principal parts. 1$^{\text{st}}$. The general theory of equations, that is to say, the collection of properties which are common to all of them. 2$^{\text{nd}}$. Their general solution, which consists in finding an expression composed of the coefficients of the given equation, and which, replacing the unknown, satisfies this equation identically, so that everything vanishes simply through cancellation. 3$^{\text{rd}}$. The solution of numerical equations, where what matters is to find particular values which satisfy an equation, all of whose coefficients are actually known and given as numbers, to as close an approximation as is desired.]

The term *équation algébrique* has *équation littérale* ('literal equation' or 'letter equation') and *équation générale* ('general equation') as common variants. Galois used all three terms, and used them synonymously.

I.8.3 The words *permutation* and *substitution*

The words *permutation* and *substitution* are of course translated as 'permutation' and 'substitution' respectively. Straightforward and natural though that is, it involves pitfalls for the unwary modern reader.

The word *permutation* is ambiguous in French, as it is in English. In English school syllabuses the word 'permutation' in the phrase 'permutations and combinations' refers to an arrangement of symbols. In undergraduate mathematics it acquires a second (and perhaps more usual) meaning as a bijection of a set to itself. Thus it is used to mean a (static) arrangement, and also to mean an act of (dynamic) rearrangement. Lagrange in [Lagrange (1770/71)] and [Lagrange (1798), 2^{nd} or 3^{rd} ed., Note XIII] used the word in both senses, but more usually dynamically (in the phrase *faire une permutation*). Cauchy in [Cauchy (1815a)] used the word *permute* as a verb in the title of his article, but used the noun *permutation* in the static sense of an arrangement in the body of his text.

The word *substitution*, on the other hand, is quite unambiguous. It always means the act of rearranging, that is to say, in modern terms a bijective mapping. This is how Cauchy defined it quite precisely in his 1815 paper cited above. Thus, referring to a function K of several variables, he wrote

> Pour indiquer cette substitution, j'écrirai les deux permutations entre parenthèses en plaçant la première au-dessus de la seconde; ainsi, par exemple, la substitution
>
> $$\begin{pmatrix} 1.2.4.3 \\ 2.4.3.1 \end{pmatrix}$$
>
> indiquera que l'on doit substituer, dans K, l'indice 2 à l'indice 1, l'indice 4 à l'indice 2, l'indice 3 à l'indice 4 et l'indice 1 à l'indice 3.
>
> [To indicate this substitution I write the two permutations between parentheses, placing the first above the second; thus, for example, the substitution
>
> $$\begin{pmatrix} 1.2.4.3 \\ 2.4.3.1 \end{pmatrix}$$
>
> will indicate that one must substitute the index 2 for the index 1, the index 4 for the index 3, the index 3 for the index 4 and the index 1 for the index 3 in K.]

In this context it should be noted that when Cauchy returned to substitutions in 1845, and wrote his many *Compte rendus* papers, of which [Cauchy (1845a)] is (or are) the first, and his long memoir [Cauchy (1845b)] (the title of which includes an interesting use of the words *arrangements*, *permutations*, and *substitutions*), he turned his two-line notation the other way up. Thus in the works of 1845 (see [Neumann (1989)] for an account of the dating of these works) we must read $\begin{pmatrix} B \\ A \end{pmatrix}$ as the substitution in which the arrangement B replaces the arrangement A.

Galois sometimes used the verb *permuter*, 'to permute' in the sense of 'to rearrange'; see, for example, Lemma II and the proofs of Lemmas III and IV of the First Memoir. Mostly, however, he used noun-forms and followed Cauchy's 1815 language, using *permutation* to mean 'arrangement' (static) and *substitution* to mean 'substitution' or 'act of rearrangement' (dynamic). Unfortunately, however, he did not use the words consistently. Sometimes he used *permutation* where he meant *substitution*—moreover, he caught himself doing this from time to time, and changed the former to the latter. The reader must be aware of the ambiguity and, where it is not immediately clear, infer the meaning from the context.

I.8.4 The word *groupe*

The word *groupe* is of course translated as 'group'. Note, however, that in the writings of Galois a group is always a group of permutations or a group of substitutions. These are not the same.

In Galois' writings a group of substitutions is a collection (non-empty goes without saying) of substitutions that is closed under composition. Since these are always substitutions of a finite number of 'letters' (the roots of a polynomial equation), closure under composition automatically implies that the identity lies in the collection and that the collection is closed under formation of inverses. Thus a *groupe de substitutions* is what we know as a group of (confusingly) permutations, a subgroup of the relevant symmetric group.

Galois also has *groupes de permutations*. A *groupe de permutations* is a collection of arrangements with the property that the collection of substitutions that change any one to any other of them is closed under composition, that is to say, is a group of substitutions. Originally these were the fundamental tools that he invented for Proposition I of the First Memoir. Gradually, though, as his thinking developed (as seen in the First Memoir, then the Second Memoir, and finally the Testamentary Letter), we see groups of substitutions becoming the principal objects of study.

Unfortunately, Galois often used the word *groupe* without specifying which kind of group he had in mind. Usually the context gives a clear indication whether the group in question is a group of permutations (arrangements) or a group of substitutions; sometimes the reader has to work quite hard to discover what was intended; and sometimes (though rarely), of course, it does not much matter.

At first Galois used the word *groupe* as an ordinary French noun meaning 'group', 'set', 'collection'. It acquired a technical meaning only through repeated use. When the academy referees read his *Premier Mémoire* they would have had to infer any special meaning of the word from the proof of Proposition I, from the first scholium (see p. 116) that follows it, and (if they were not already stymied) from the regular use of it later in the paper. At that time Galois had not explained its meaning. It was only at the final revision on the eve of the fatal duel that he added his famous explicit definition (**f. 3b**, p. 114):

> Les substitutions sont le passage d'une permutation à l'autre.
>
> La permutation d'où l'on part pour indiquer les substitutions est toute arbitraire [...]
>
> [...]

> Donc si dans un pareil groupe on a les substitutions S et T, on est sûr d'avoir la substitution ST.
>
> [Substitutions are the passage from one permutation to another.
>
> The permutation from which one starts in order to indicate substitutions is completely arbitrary [...]
>
> [...]
>
> Therefore, if in such a group one has the substitutions S and T, one is sure to have the substitution ST.]

He added this in the margin alongside Proposition I of the *Premier Mémoire*, with the instruction to print it in the introductory section, which is where it correctly appears in all editions previous to this one.

The point made in the foregoing paragraph is illustrated by comparison with the word *ensemble* ('set', 'collection'). Galois used this word, but it did not acquire a technical meaning in his writings. It remained an informal word. Compare the words *groupe* and *ensemble* in passages where they occur together: in the deleted sentence before the last paragraph of the definition of *groupe* in the *Premier Mémoire* (**f. 3 b**, p. 114), in Dossier 15, **f. 82 a**, **f. 83 b** and **f. 84 a**, and in the first example to Proposition I of the *Premier Mémoire* (**f. 3 b**, p. 114, **f. 54 b** in Dossier 6, and **f. 87 a**, **f. 88 a** in Dossier 16).

It has been suggested to me that Galois may have acquired his word *groupe* from the lovely preface by Poinsot to the 1826 edition of Lagrange's treatise on numerical solution of equations (cited above, p. 20). Poinsot used the word informally at first, and although, as he continued to develop the ideas he was expounding there, it gradually acquired some of the characteristics of a technical term, the context is rather different from that of Galois. Poinsot's are groups of roots of an equation, not of permutations or substitutions of roots. In modern terms we can recognise them as being akin to blocks of imprimitivity of a transitive permutation group acting on the set of roots; indeed, in anachronistic terms, once the relevant Galois theory is in place, that is precisely what they are. That this is where Galois got the word is certainly a possibility. I am inclined to doubt it, though. And I doubt very much that it is where he got his concept—there is nothing like his notions of *groupe* in Poinsot's preface. Reading what Galois wrote I get a strong impression that, as I have explained above, he began by simply using the word *groupe* as an ordinary and convenient noun. He could as well have chosen some other collective noun, except that *groupe* is simple and direct and has the associated verb *grouper*. It seems extremely unlikely that he owed it to Poinsot.

I.8.5 The word *semblable*

The word *semblable* is naturally translated as 'similar'. At many points it has a technical meaning. Most often it occurs in plural form as an adjective qualifying *groupes*. Thus for example, in the discussion following the statement of Proposition VII of the

First Memoir, and in **f. 84 a** (Dossier 15), if Γ is a group of permutations (arrangements) and S is a substitution, then ΓS is a group of permutations and Γ, ΓS are *groupes semblables*. (I have distorted the notation here, using Γ where Galois used G, because I want to emphasize the distinction between a group of permutations and a group of substitutions.) Note that if G is the group of substitutions corresponding to the group Γ of permutations then the group of substitutions corresponding to ΓS is $S^{-1}GS$.

There are a few points where the adjectival phrase *semblables et identiques* is used to qualify *groupes*. This should be understood as follows: there is a group Γ of permutations (arrangements) with group G of substitutions; this contains a group Δ of permutations with group H of substitutions; what is meant is that H is a normal subgroup of G. The point is this. Let $S_1, S_2, \ldots, S_m$ be right coset representatives for H in G, so that $G = HS_1 \cup HS_2 \cup \cdots \cup HS_m$ (a disjoint union). Then $\Gamma = \Delta S_1 \cup \Delta S_2 \cup \cdots \cup \Delta S_m$, a union of *groupes semblables*. The group ΔS_i of permutations (arrangements) corresponds to the group $S_i^{-1}HS_i$ of substitutions. Given that H is a normal subgroup of G these are identical; hence *semblables et identiques*, 'similar and identical'. The groups of permutations are similar, their groups of substitutions are identical. That this reading is likely to be robust is confirmed by a passage on **f. 55 b** (Dossier 7), where Galois changed

> Il faut donc que le groupe G se partage en p groupes H semblables et identiques
>
> [It is therefore necessary that the group G be partitioned into p similar and identical groups H.

to

> Il faut donc que le groupe G se partage en p groupes H semblables et dont les substitutions soient les mêmes
>
> [It is therefore necessary that the group G be partitioned into p groups H that are similar and of which the substitutions are the same]

The *décomposition propre* of the *Lettre testamentaire*, **f. 8 a** is the same thing. Also, in the Second Memoir, for example in **f. 37 b**, **f. 39 a**, **f. 39 b**, Galois used *groupes conjugués* ('conjugate groups') in what appears (from the context) to be a similar sense, that is to say, for groups of permutations contained in a larger group and such that their groups of substitutions are all equal and form a normal subgroup of the group of substitutions of the large one.

I.8.6 The word *primitif*

The word *primitif* is naturally translated as 'primitive'. It is a technical term for which Galois gave a definition in his 'Analyse d'un Mémoire' published in Férussac's *Bulletin*, April 1830, in **f. 8 b** of the *Lettre testamentaire*, and at one or two other points (sometimes implicitly). The paper [Neumann (2006)] devotes some 50 printed pages to the word. I do not propose to enter into such detail here. It must suffice to remind the reader that in this context a *groupe primitif* should usually be thought of in modern terms as a quasi-primitive permutation group, that is to say, a permutation group with the property that every non-trivial normal subgroup is transitive; an *équation primitive* is then an equation or polynomial whose Galois group is quasi-primitive in its action on the set of roots.

I.8.7 Other words and phrases

In addition to (semi-)technical terms discussed above there are other words and constructions that Galois used which are not easy to translate. It is easy enough to see what he meant, but finding a way of saying it in English using similar words and constructions is not easy. Here are some common examples:

- I have chosen to translate *ensemble* as 'collection' because 'set' has a modern technical meaning, and Galois used the word as an ordinary noun, not as a technical term.
- I have chosen to translate *équation proposée* literally as 'proposed equation'. Nowadays we would more naturally write 'given equation' but most writers of the time used *équation proposée*, in preference to *équation donnée* (which, however, one does see from time to time).
- Galois made extensive use of the construction *x étant*. Mostly I have used the ugly literal translation 'x being', but more common (and more agreeable) English usage is 'where x is'.
- Constructions such as *Remarquons que, Prenons*, etc., might be translated literally as 'We note that', 'We take', etc. In French however, they indicate an imperative, and so they should be translated into the imperative mood 'Note that', 'Take', etc., that is common also in mathematical English.
- Constructions such as *on obtient* are often translated into the passive rather than 'one obtains'.
- The word *caractère* should naturally mean 'character' or 'characteristic', but Galois often used it to mean 'property' or 'condition'.

I.8.8 Glossary

For the convenience of the reader I summarise here some of the discussion above, and add a few further words to the dictionary.

Analyse is naturally translated as 'analysis', but with a meaning that is close to algebra in our context.

Analyste is naturally translated as ‘analyst’, but the meaning is closer to ‘algebraist’ or ‘pure mathematician’ (compare *géomètre*).

Équation is naturally translated as ‘equation’, but very often is used where we would use ‘polynomial’.

Degré is naturally translated as ‘degree’. Mostly Galois used it in exactly the same way as it is used nowadays as in degree of an equation or a polynomial, degree of a radical (or algebraic number), degree of a substitution group (as the number n such that the group is a subgroup of $\mathrm{Sym}(n)$), degree of an algebraic surface. Note, however, that in [Cauchy (1815a), p. 13] *le degré* [*d'une*] *substitution* is defined to be what we would call its order: the degree of S is the least m such that S^m is the identity. There are a few passages where Galois seems to have this usage, or something similar to it, in mind—see **f. 84 b** and **f. 91 a**, for example (pp. 290, 314)—but there are discrepancies in that Galois could there be intending the number of letters permuted.

Géomètre is naturally translated as ‘geometer’, but the meaning is closer to ‘pure mathematician’ (compare *analyste*).

Groupe, sometimes groups of permutations (static arrangements), sometimes groups of substitutions.

Groupe partiel, is naturally translated as ‘partial group’. Think of the modern terms ‘subgroup’ and ‘coset’. In some contexts one is appropriate, in others the other.

Groupe sousmultiple, naturally translated as ‘submultiple group’, to be thought of as ‘subgroup’. Galois also used the word *diviseur* ‘divisor’.

Mémoire is naturally translated as ‘memoir’. This works well in 19th century English, though ‘article’ or ‘paper’ would be more common now.

Ordre is naturally translated as ‘order’. Note, however, that in the context ‘la substitution sera de l'ordre p’ it usually means the number of letters moved by the substitution.

Période is naturally translated as ‘period’. Note, however, that in the context ‘substitution dont la période sera de p termes’ it means what is now called order—the substitution will have order p.

Permutation is naturally translated as ‘permutation’ but with the meaning of an arrangement (static).

Substitution is naturally translated as ‘substitution’, meaning an act of rearranging (dynamic); unfortunately, nowadays we use the word ‘permutation’.

Substitution circulaire is naturally translated as ‘circular substitution’. It would be very dangerous to follow one’s instinct and use ‘cyclic substitution’ (or ‘cyclic permutation’) because that is not what it appears to mean. In the First Memoir, in passages about equations of prime degree p, the term refers to the whole cyclic group generated by a p-cycle. In the Second Memoir it is less clear what is meant, but at one point (see p. 178) Galois used the term to refer to the substitutions of prime order that lie in the (unique, as it happens) normal abelian subgroup of his primitive permutation group of prime-squared degree. At that point these have p p-cycles.

The usage of this term by Galois seem to differ considerably from the meaning carefully defined by Cauchy in [Cauchy (1815a), p. 17], where a *substitution circulaire* is very clearly defined as a cyclic substitution of the indices (letters, points) that it does not fix.

Transformer is translated as ‘to transform’. It occurs a few times in the Second Memoir, where its meaning is what I can best describe in modern language as ‘transform by conjugation’.

I.9 A ‘warts and all’ transcription

As has been indicated above, the French is intended as a ‘warts-and-all’ transcription. What this means is that I have sought to reproduce the manuscript as accurately as possible in print, with all its crossings-out, emendations, additions and quirks of writing. In particular, the crossed-out material, except where it consists of no more than one or two illegible letters, has been included in its proper place. (Most, but not all, of this was transcribed by Robert Bourgne and appears on the left hand pages of [B & A (1962)].) Some of the crossed-out material was simply abandoned as Galois proceeded to his next phrase or sentence. But sometimes a word or two here and there was retained and re-used. Thus, for example, near the bottom of **f. 39 a** in the Second Memoir there was a passage that might have read

> [...]. Ce n’est point $p^2 - p$, puisque le groupe G serait non primitif.
> Mais il faut que les substitutions [...]

Then three words from the beginning of the second paragraph were deleted, the words ‘que les’ were retained and incorporated into the revised text, the word ‘substitutions’ was changed to ‘permutations’, and new text was inserted at the end of the previous paragraph, so as to read

> [...]. Ce n’est point $p^2 - p$, puisque le groupe G serait non primitif. Si donc dans le groupe G on ne considère que les permutations [...]

without a paragraph break. Independently of all this the words ‘dans ce cas’ were inserted so that the final text reads

> [...]. Ce n'est point $p^2 - p$, puisque dans ce cas le groupe G serait non primitif. Si donc dans le groupe G on ne considère que les permutations [...]

It should be noted also that there was a false start, immediately broken off, to the second paragraph. Also, it is quite possible that at the first pass Galois stopped after 'Mais il faut'.

The following notes are intended to make the phrase 'a warts and all transcription' a little more precise.

- Misprints, mis-spellings and infelicities of punctuation have been retained; note that mis-spellings include unconventional use or non-use of diacritical marks. Often accents, especially acute accents are missing; where 'é' follows 't', however, the accent could be swallowed up in the crossbar of the 't' and where it is unclear I have assumed that it is there. Some of these may be not so much mis-spellings as dated usage, such as 'tems' for 'temps'. Others may be not so much mis-spellings as haste—a circumflex accent might easily emerge like a grave accent late on a pre-duel evening.

  The lists below are unlikely to be complete—I did not start compiling them until well into the work. I hope, however, that they may give the reader some idea of the problem. Now that images of the manuscripts have (from June 2011) become available through web-publication in digitised form (see [Galois (2011)], p. 392), these and other infelicities are capable of being checked. I believe that the reader will find

  - 'a' for à',
  - 'algèbrique' or 'algebrique' for 'algébrique',
  - 'appêllerons' for 'appellerons',
  - 'appèle' for 'appelle',
  - 'arrèter' for 'arrêter',
  - 'bientòt' for 'bientôt',
  - 'celà' for 'cela',
  - 'complementaire' for 'complémentaire',
  - 'completer' for 'compléter',
  - 'connait' for 'connaît',
  - 'consequent' for 'conséquent',
  - 'coté' for 'côté',
  - 'dégré' for 'degré',
  - 'doît' for 'doit',
  - 'dù' for 'dû',

- ‘ecrire’ for ‘écrire’,
- ‘equation’ for ‘équation’,
- ‘ètre’ or ‘etre’ for ‘être’,
- ‘eùt’ for ‘eût’,
- ‘evidemment’ for ‘évidemment’,
- ‘éxemple’ for ‘exemple’,
- ‘éxige’ for ‘exige’,
- ‘frequent’ for ‘fréquent’,
- ‘gachis’ for ‘gâchis’,
- ‘general’ for ‘général’,
- ‘geometre’ for ‘géomètre’,
- ‘guere’ for ‘guère’,
- ‘intéret’ for ‘intérêt’,
- ‘meme’ or ‘mème’ for ‘même’,
- ‘numeriques’ for ‘numériques’,
- ‘ou’ for ‘où’,
- ‘paraitrait’ for ‘paraîtrait’,
- ‘partageàt’ for ‘partageât’,
- ‘periode’ for ‘période’,
- ‘plutot’ for ‘plutôt’,
- ‘précedemmant’ for ‘précédemmant’,
- ‘premiérement’ for ‘premièrement’,
- ‘prevoir’ for ‘prévoir’,
- ‘pùt’ for ‘pût’,
- ‘rebùte’ for ‘rebute’,
- ‘reconnaitre’ for ‘reconnaître’,
- ‘regles’ for ‘règles’,
- ‘remplacant’ for ‘remplaçant’,
- ‘reponde’ for ‘réponde’ and ‘repondre’ for ‘répondre’,
- ‘resolues’ for ‘résolues’,
- ‘resultat’ for ‘résultat’ (and in plural),
- ‘siécle’ or ‘siecle’ for ‘siècle’,
- ‘symmétrique’, ‘symmetrique’ or ‘symmettrique’ for ‘symétrique’,

  - ‘tems’ for ‘temps’ [but perhaps this is simply a case of old spelling],
  - ‘tète’ for ‘tête’,
  - ‘théoreme’ for ‘théorème’;

  note also that there is little consistency here;

- inconsistencies in hyphenation, such as
  - ‘c’est à dire’ *v.* ‘c’est-à-dire’.
  - ‘non-primitif’ *v.* ‘non primitif’,
  - ‘peut être’ *v.* ‘peut-être’.
- Unconventional but consistent usages, such as ‘de suite’ for ‘tout de suite’, have been retained.
- Often Galois clearly ran two words together, as in ‘àmoins’, ‘cequi’, ‘delà’, ‘demême’, ‘deplus’, ‘enaura’, ‘enfonction’, ‘ensorte’, ‘entout’, ‘entant’, ‘desuite’ (as in ‘ainsi desuite’), ‘lemoyen’, ‘oubien’, ‘parconséquent’, ‘quelque’ for ‘quel que’ (though of course ‘quelque’ also exists as a genuine word), ‘yavait’;
- contrariwise sometimes Galois clearly would split a word, as in ‘la quelle’ for ‘laquelle’ or ‘les quelles’ for ‘lesquelles’, ‘en suite’ for ‘ensuite’, ‘puis que’ for ‘puisque’, ‘si non’ for ‘sinon’;

Some of these infelicities are sporadic, some are more systematic, none are greatly disturbing.

The notation that Galois used was generally clear and conventional, though the Second Memoir and a few other manuscripts have some unconventional usages:

- In the Second Memoir Galois used a clearly and carefully written . (rather than ,) to separate indices, as in $a_{1.0}$, etc.; moreover, within the text a dot often stands for a comma—but this usage has not been retained as it would be too confusing, especially since the manuscripts also contain a number of random and otiose non-punctuating dots which, one may conjecture, Galois may have made by touching the paper lightly with his pen on finishing a word or formula;
- In the Second Memoir (and several other places) Galois almost always, with just a very few exceptions, wrote his 2nd order inferiors directly below the 1st order inferiors as in $a_{\substack{k\\1}}$, etc.;
- Galois used four or more dots $\cdot\cdot\cdot\cdot$ for ellipsis.

In the manuscripts such labels as ‘Lemme’, ‘Théorème’, ‘Démonstration’ are written in the same style as the main text, as are the statements that they label. In his copy Chevalier doubly underlined them; Liouville used SMALL CAPITALS for the main labels and *italics* for subsidiary ones; Bourgne & Azra used SMALL CAPITALS. Liouville used quotation marks to indicate the status of statements of theorems, lemmas, etc., Picard italicised them in the conventional way, Bourgne & Azra followed Galois in not distinguishing. For the present warts-and-all edition I have chosen to follow Galois in the French, but to use modern conventions in the English.

In some of his writings Galois indented the first lines of his paragraphs, in others he did not. I have not studied the phenomenon carefully, but I have an impression that it is the later writings that have indentations. If so, and if this is systematic, then it could help to date the various items. Where paragraphs are not indented their beginnings can usually be deduced from the fact that the previous line is not full and that there is sometimes a very thin space between paragraphs. The English translation, which does have paragraph indentations, should help to clarify what is going on in the French.

Often Galois used a thin horizontal line to indicate that he had come to a full cadence or finished a passage of writing. Many of these seem to me to be significant, and I have tried to reproduce them as faithfully as possible in the transcription.

I.10 The transcription: editorial conventions

In physical descriptions of the manuscripts $a\,\mathrm{cm} \times b\,\mathrm{cm}$ gives width (East–West, direction of lines of writing) first, then depth (North–South). In his emendations and other adjustments to his writings Galois made heavy use of the two-dimensional nature of a page. In my transcriptions I have tried to produce something that reflects the order and the disorder of the manuscripts. The linearity of print makes that well-nigh impossible of course. There are, however, some techniques that help. The following conventions differ from, but are similar in concept to, some of those used by Bourgne & Azra (see [B & A (1962), p. xxxiii]):

- brackets ^text^ indicate emendations or afterthoughts inserted above the line;
- brackets ⌄text⌄ indicate emendations or afterthoughts inserted below the line;
- brackets ⌜text⌝ indicate afterthoughts inserted on the line of text;
- asterisks used as brackets *text* indicate afterthoughts written into the margin (the left margin—there is no right margin).

Note that Galois himself used asterisks to indicate footnotes and also some of his marginal additions. These are here rendered as ✳ or ✳ .

It has been impossible to mimic the deletions in the manuscript as closely as I would have liked. Although many of them are done with a single line, sometimes Galois used a very heavy line, sometimes a wavy line, sometimes he cross-hatched. Two lines ~~thus~~ simply indicate heavier crossing-out than one ~~thus~~. Where something is crossed out and replaced (usually with a word or two above the line) it is rendered without a space thus: ~~rendered~~^transcribed^. A space, small though it may be, as in ~~rendered~~ ^A space^, indicates that the insertion was not a replacement for the deleted material. Or at least, that that is my belief.

Where I have been unable to decipher a word I have used [?], [??], [???] or [????], the number of question marks indicating the probable number of illegible syllables.

Most, but not quite all, of the crossed out material has been translated. This is partly as a service to the reader, partly to help keep the English running properly in tandem with the French on the opposite page.

Just as it is impossible to make a perfect translation, so it is impossible to transcribe a manuscript into print entirely faithfully. Choices have had to be made. I have sought to get as close to the original as I could, but it would not have been helpful, even had it been possible, to reproduce in type the exact layout on the page, such as line-breaks or replacement above (and sometimes below) the line, or in the margin, of crossed-out words and phrases. Nevertheless, I hope that this transcription will be found to be satisfactorily close to the original. If nothing else, the retention of infelicities of spelling and grammar, and the many indications of emendations and afterthoughts, should keep at the front of our mind the fact that we are dealing here with mainly unpolished manuscripts by an untrained young genius of another age.