The QGM Master Class Series

Edited by Jørgen Ellegaard Andersen, Henning Haahr Andersen, Nigel Hitchin, Maxim Kontsevich, Robert Penner and Nicolai Reshetikhin

The Centre for Quantum Geometry of Moduli Spaces (QGM) in Aarhus, Denmark, focuses on collaborative cutting-edge research and training in the quantum geometry of moduli spaces. This discipline lies at the interface between mathematics and theoretical physics combining ideas and techniques developed over the last decades for resolving the big challenge to provide solid mathematical foundations for a large class of quantum field theories.

As part of its mission, QGM organizes a series of master classes each year continuing the tradition initiated by the former Center for the Topology and Quantization of Moduli Spaces. In these events, prominent scientists lecture on their research speciality starting from first principles. The courses are typically centered around various aspects of quantization and moduli spaces as well as other related subjects such as topological quantum field theory and quantum geometry and topology in more general contexts.

This series contains lecture notes, textbooks and monographs arising from the master classes held at QGM. The guiding theme can be characterized as the study of geometrical aspects and mathematical foundations of quantum field theory and string theory.

Robert C. Penner

Decorated Teichmüller Theory

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Dedicated to my mother
Beverly Preston Penner
in memoriam

Prologue

This book is intended to be a self-contained and elementary presentation from first principles of a theory developed by the author and collaborators over the last 25 years. It has arisen from lecture notes for a master's class on decorated Teichmüller theory taught at Aarhus University during August, 2006, under the aegis of the Center for the Topology and Quantization of Moduli Spaces, which has evolved into the Center for the Quantum Geometry of Moduli Spaces. It is an honor to present this inaugural volume in what is hoped will be a stimulating series based on master's classes presented at the center.

It is a great pleasure to thank my colleague, collaborator and friend Jørgen Ellegaard Andersen for organizing the master's class and arranging my original visit to Aarhus, which is now my principal academic home. Let me also thank the other participants in the class, who made many valuable and sometimes critical comments, including Marcel Bökstedt, Bill Browder, Niels Gammalgaard, Magnus Lauridsen, especially Gregor Masbaum, Guillaume Theret, Rasmus Villemoes and Yannis Vlassopoulos. Further thanks are due to Alex Bene, Rinat Kashaev and Dylan Thurston for useful discussions.

One goal of this monograph is to present global affine coordinates on decorated Teichmüller spaces as well as natural cell decompositions of these spaces, which are necessary for the quantization of Teichmüller spaces and for the simplest geometric occurrences of cluster algebras. However, there are many further applications and extensions of these basic ingredients discussed here as well, for instance, to algebraic number theory, harmonic analysis, profinite and pronilpotent versions of surfaces, the topology of Riemann's moduli spaces, topological and conformal field theories and computational biology.

Foreword

One can argue that the general idea of a moduli space, viewed in its philosophical dimension, lies at the very heart of modern scientific thinking. A moduli space of whatever nature is an incarnation of the "manifold of possibilities", be it the phase space of Solar System, the Hilbert space of quantum state vectors of a hydrogen atom, or, as in one of the papers of R. Penner and M. Waterman of 1993, the space of secondary structures of an RNA molecule.

At the next stage of scientific reflection, "scientific laws" are introduced further constraining time evolution of a system in its phase space, or specifying a measure on it, its subsets and regions, say "equilibrium points" or "attractors", that are more directly responsible for the explanation of observable phenomena.

And as a final check and a hopeful triumph for a theory, observable phenomena and/or results of controlled experiments are seen to fit (or not to fit ...) their expected patterns in the relevant moduli space, the manifold of possibilities. In pure mathematics, "moduli" of the objects of a given type are associated with the image of a space of parameters on which such objects can depend. Historically early modern examples include Grassmannian spaces of linear subspaces, upper complex half-plane as parameter space for elliptic curves, and brilliant generalizations and innovations due to Riemann. In the second half of the XX century, such thinkers as Alexander Grothendieck and William Thurston contributed their very different visions to the development of this general idea.

The book "Decorated Teichmüller Theory" by R. C. Penner is a beautifully presented survey of some of the most important work of the last two decades dedicated to the moduli of two-dimensional geometric objects, complex and/or Riemannian (metric) surfaces, compact or, more often, satisfying appropriate restrictions on boundaries (finite number of points, or a union of small horocycles etc.). One of the dominating characteristics of Penner's approach is a rich and visually appealing representation of surfaces as embedded in three-dimensional Minkowski space and interacting there with piecewise linear structures such as triangulations, embedded graphs etc., described geometrically in terms of hyperbolic lengths, flatness and combinatorics. One can consider this construction as a descendant of classical Dirichlet–Voronoi methods in the theory of lattices.

Such metric characteristics related to a single surface become then parameters on the moduli space of all such surfaces, and, as Penner's body of research abundantly shows, they are so successfully constructed, that a host of known structures can be very efficiently described in their context. As examples, one can name the Weil–Petersson forms, the Thurston boundary of Teichmüller space, the action of the mapping class group on the decorated Teichmüller space, useful and beautiful cell decompositions, and many more. A version of Penner's cell decomposition was used by Maxim Kontsevich in his spectacular proof of Ed Witten's conjectures on the intersection numbers of moduli spaces.

Returning to the general picture of "manifolds of possible" in science, one must now mention that hyperbolic surfaces embedded in a Minkowski space can be imagined as "world sheets" of quantum strings. This image motivated much interesting research and interaction between mathematicians and string theorists in the physical community inspired by Ed Witten. This interaction, to which Penner contributed a series of important insights and results, underlies one of the many aspects that will attract a student or a researcher to study this book.

Yuri I. Manin