Introduction

This set of notes explores some of the links between occupation times and Gaussian processes. Notably they bring into play certain isomorphism theorems going back to Dynkin [4], [5] as well as certain Poisson point processes of Markovian loops, which originated in physics through the work of Symanzik [26]. More recently such Poisson gases of Markovian loops have reappeared in the context of the "Brownian loop soup" of Lawler and Werner [16] and are related to the so-called "random interlacements", see Sznitman [27]. In particular they have been extensively investigated by Le Jan [17], [18].

A convenient set-up to develop this circle of ideas consists in the consideration of a finite connected graph E endowed with positive weights and a non-degenerate killing measure. One can then associate to these data a continuous-time Markov chain $\overline{X}_t$, $t \geq 0$, on E, with variable jump rates, which dies after a finite time due to the killing measure, as well as

$$\text{the Green density } g(x, y), \;\; x, y \in E, \tag{0.1}$$

(which is positive and symmetric),

$$\text{the local times } \overline{L}_t^x = \int_0^t 1\{\overline{X}_s = x\}\, ds, \;\; t \geq 0, x \in E. \tag{0.2}$$

In fact $g(\cdot, \cdot)$ is a positive definite function on $E \times E$, and one can define a centered Gaussian process φ_x, $x \in E$, such that

$$\operatorname{cov}(\varphi_x, \varphi_y)(= E[\varphi_x \varphi_y]) = g(x, y), \;\; \text{for } x, y \in E. \tag{0.3}$$

This is the so-called Gaussian free field.

It turns out that $\frac{1}{2}\,\varphi_z^2$, $z \in E$, and $\overline{L}_\infty^z$, $z \in E$, have intricate relationships. For instance Dynkin's isomorphism theorem states in our context that for any $x, y \in E$,

$$\big(\overline{L}_\infty^z + \frac{1}{2}\,\varphi_z^2\big)_{z \in E} \;\;\text{under } P_{x,y} \otimes P^G \tag{0.4}$$

has the "same law" as

$$\frac{1}{2}\,(\varphi_z^2)_{z \in E} \;\;\text{under } \varphi_x \varphi_y\, P^G, \tag{0.5}$$

where $P_{x,y}$ stands for the (non-normalized) h-transform of our basic Markov chain, with the choice $h(\cdot) = g(\cdot, y)$, starting from the point x, and P^G for the law of the Gaussian field φ_z, $z \in E$.

Eisenbaum's isomorphism theorem, which appeared in [7], does not involve h-transforms and states in our context that for any $x \in E$, $s \neq 0$,

$$\big(\overline{L}^z_\infty + \tfrac{1}{2}\,(\varphi_z + s)^2\big)_{z\in E} \quad \text{under } P_x \otimes P^G \tag{0.6}$$

has the "same law" as

$$\Big(\frac{1}{2}\,(\varphi_z + s)^2\Big)_{z\in E} \quad \text{under} \Big(1 + \frac{\varphi_x}{s}\Big)\, P^G. \tag{0.7}$$

The above isomorphism theorems are also closely linked to the topic of theorems of Ray–Knight type, see Eisenbaum [6], and Chapters 2 and 8 of Marcus–Rosen [19]. Originally, see [13], [21], such theorems came as a description of the Markovian character in the space variable of Brownian local times evaluated at certain random times. More recently, the Gaussian aspects and the relation with the isomorphism theorems have gained prominence, see [8], and [19].

Interestingly, Dynkin's isomorphism theorem has its roots in mathematical physics. It grew out of the investigation by Dynkin in [4] of a probabilistic representation formula for the moments of certain random fields in terms of a Poissonian gas of loops interacting with Markovian paths, which appeared in Brydges–Fröhlich–Spencer [2], and was based on the work of Symanzik [26].

The Poisson point gas of loops in question is a Poisson point process on the state space of loops on E modulo time-shift. Its intensity measure is a multiple $\alpha\mu^*$ of the image μ^* of a certain measure μ_{rooted}, under the canonical map for the equivalence relation identifying rooted loops γ that only differ by a time-shift. This measure μ_{rooted} is the σ-finite measure on rooted loops defined by

$$\mu_{\text{rooted}}(d\gamma) = \sum_{x\in E} \int_0^\infty Q^t_{x,x}(d\gamma)\, \frac{dt}{t}, \tag{0.8}$$

where $Q^t_{x,x}$ is the image of $1\{X_t = x\}\, P_x$ under $(X_s)_{0\le s\le t}$, if $X_{\boldsymbol{.}}$ stands for the Markov chain on E with jump rates equal to 1 attached to the weights and killing measure we have chosen on E.

The random fields on E alluded to above, are motivated by models of Euclidean quantum field theory, see [11], and are for instance of the following kind:

$$\langle F(\varphi)\rangle = \int_{\mathbb{R}^E} F(\varphi)\, e^{-\frac{1}{2}\mathcal{E}(\varphi,\varphi)} \prod_{x\in E} h\Big(\frac{\varphi_x^2}{2}\Big)\, d\varphi_x \Big/ \int_{\mathbb{R}^E} e^{-\frac{1}{2}\mathcal{E}(\varphi,\varphi)} \prod_{x\in E} h\Big(\frac{\varphi_x^2}{2}\Big)\, d\varphi_x \tag{0.9}$$

with

$$h(u) = \int_0^\infty e^{-vu}\, d\nu(v),\; u \ge 0,\; \text{ with } \nu \text{ a probability distribution on } \mathbb{R}_+,$$

and $\mathcal{E}(\varphi, \varphi)$ the energy of the function φ corresponding to the weights and killing measure on E (the matrix $\mathcal{E}(1_x, 1_y)$, $x, y \in E$ is the inverse of the matrix $g(x, y)$, $x, y \in E$ in (0.3)).

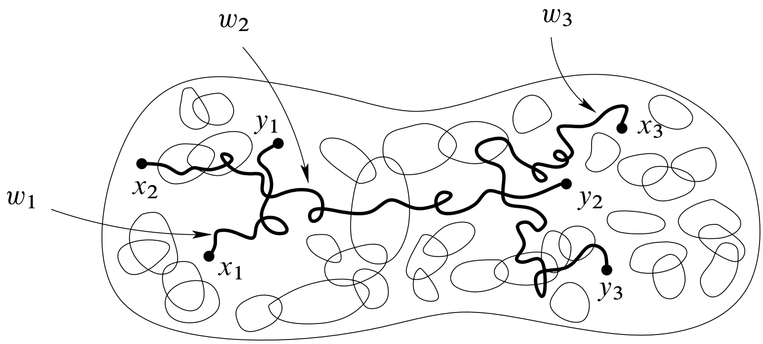

Figure 0.1. The paths $w_1, \dots, w_k$ in E interact with the gas of loops through the random potentials.

The typical representation formula for the moments of the random field in (0.9) looks like this: for $k \geq 1$, $z_1, \dots, z_{2k} \in E$,

$$\langle \varphi_{z_1} \dots \varphi_{z_{2k}} \rangle = \sum_{\substack{\text{pairings} \\ \text{of } z_1, \dots, z_{2k}}} \frac{P_{x_1,y_1} \otimes \cdots \otimes P_{x_k,y_k} \otimes \mathbb{Q}\big[e^{-\sum_{x \in E} v_x (\mathcal{L}_x + \bar{L}^x_\infty(w_1) + \cdots + \bar{L}^x_\infty(w_k))}\big]}{\mathbb{Q}\big[e^{-\sum_{x \in E} v_x \mathcal{L}_x}\big]}, \tag{0.10}$$

where the sum runs over the (non-ordered) pairings (i.e. partitions) of the symbols $z_1, z_2, \dots, z_{2k}$ into $\{x_1, y_1\}, \dots, \{x_k, y_k\}$. Under $\mathbb{Q}$ the $v_x, x \in E$, are i.i.d. ν-distributed (random potentials), independent of the $\mathcal{L}_x$, $x \in E$, which are distributed as the total occupation times (properly scaled to take account of the weights and killing measure) of the gas of loops with intensity $\frac{1}{2}\,\mu$, and the P_{x_i,y_i}, $1 \leq i \leq k$ are defined just as below (0.4), (0.5).

The Poisson point process of Markovian loops has many interesting properties. We will for instance see that when $\alpha = \frac{1}{2}$ (i.e. the intensity measure equals $\frac{1}{2}\,\mu$),

$$\begin{aligned} &(\mathcal{L}_x)_{x \in E} \text{ has the same distribution as } \tfrac{1}{2}\,(\varphi_x^2)_{x \in E}, \text{ where} \\ &(\varphi_x)_{x \in E} \text{ stands for the Gaussian free field in (0.3).} \end{aligned} \tag{0.11}$$

The Poisson gas of Markovian loops is also related to the model of random interlacements [27], which loosely speaking corresponds to "loops going through infinity". It appears as well in the recent developments concerning conformally invariant scaling limits, see Lawler–Werner [16], Sheffield–Werner [24]. As for random interlacements, interestingly, in place of (0.11), they satisfy an isomorphism theorem in the spirit of the generalized second Ray–Knight theorem, see [28].