Preface

> "To every loving, gentle-hearted friend,
> to whom the present rhyme is soon to go
> so that I may their written answer know (…)"
>
> Translated from
> *A ciascun'alma presa e gentil core, La Vita Nuova*
> Dante Alighieri, 1295

Local Moufang conditions

In two famous papers [16], [17], Fong and Seitz showed that all finite Moufang generalized polygons were classical or dual classical. In fact, they obtained this result in group theoretical terms (classifying finite split BN-pairs), but Tits remarked the simple geometrical translation. And of course, the converse was already well known. In a search for a synthetic "elementary" proof of the Fong–Seitz result for the specific case of generalized quadrangles (which is the central and most difficult part in [16], [17]), Payne and J. A. Thas noticed that when one looks at the group generated by all root-elations and dual root-elations which stabilize a given point of a Moufang quadrangle, the group fixes all lines incident with that point, and acts sharply transitively on its opposite points. Let us call a point with this property an *elation point*, and a generalized quadrangle with such a point an *elation generalized quadrangle*. Kantor noticed in the early 1980s that, starting from a group with a suitable family of subgroups satisfying certain properties, one can construct an elation quadrangle from this data in a natural way, such that the group acts as an elation group. This process can be easily reversed, so as to obtain such group theoretical data starting from any elation quadrangle. This observation is the precise analogon of the fact that, in a Moufang projective plane, any line is a translation line, and when one singles out the definition of translation line and translation plane, one can also translate the situation in group theoretical terms to a group with certain subgroups, etc. In that case, one obtains a group of order n^2 with a family of $n + 1$ subgroups of order n, two by two trivially intersecting (in the infinite case, one has to require that the product of any two of these subgroups equals the entire group and that the subgroups cover the group). And conversely, starting from such group theoretical data, one readily reconstructs a translation plane for which the group acts as a translation group. The essential difference in this correspondence between planes and quadrangles is that, in the planar case, the translation group necessarily is abelian, and this is not so for elation groups of generalized quadrangles. In the planar case, this property allows one to define a "kernel", which is some skew field over which the translation group naturally becomes a vector space. In quadrangular theory, one has to *assume* that the elation group is abelian to obtain a similar notion of kernel, and then again, the abelian elation group can be seen as a vector space. In fact, one also assumes that the quadrangle is finite, since there are some nontrivial obstructions when passing to the infinite case.

$$\begin{tikzcd}
\text{Planes} \arrow[r] \arrow[d] & \text{Translation planes} \arrow[d] \\
\text{Quadrangles} \arrow[r] & \text{Elation quadrangles}
\end{tikzcd}$$

In this monograph, we will focus on general finite elation quadrangles, so without the commutativity assumption on the group. In the commutative case a rich theory is available, and we refer the reader to [59] and the references therein for the (many) details. Another basic difference with the nonabelian case is that an abelian elation group is unique (both for planes and quadrangles). That is, there can only be at most one ("complete", that is, transitive on the appropriate point set) abelian elation group for a given line in a projective plane or point in a generalized quadrangle, and it necessarily is elementary abelian (in the finite case). As we will see in the present notes, this fact is not true for general elation quadrangles. We will encounter examples which admit different (t-maximal) elation groups with respect to the same elation point, and they even can be nonisomorphic. (As a by-product, we will construct the first infinite class of translation nets with similar properties.) Also, in the planar case and the abelian quadrangular case, any i-root and dual i-root involving the translation line or the elation point is Moufang, and the unique t-maximal elation group is generated by the Moufang elations. In general, such properties do not hold for elation quadrangles. We will obtain the first examples of finite elation quadrangles for which not every (dual) i-root involving the elation point is Moufang.

So we first have to handle these standard structural questions as a set up for the theory.

After Kantor's observations, many infinite classes of finite generalized quadrangles were constructed as elation quadrangles, through the identification of "Kantor families" in appropriate groups. Moreover, up to a combination of point-line duality and Payne integration, every known finite generalized quadrangle *is* an elation quadrangle. This observation lies at the origin of the need for a structural theory for elation quadrangles, which appears to be lacking in the literature. In fact, most of the foundations can be found in Chapters 8 and 9 of [46], and that's about it.

In the present book, I hope to fill up this gap.

Further outline

Let me briefly outline the contents of this book besides what was already mentioned.

First of all, let me mention that the three basic references on generalized quadrangles are the monographs "Finite Generalized Quadrangles" [44], [46]; "Symmetry in Finite Generalized Quadrangles" [68] and "Translation Generalized Quadrangles" [59] (on elation quadrangles with an abelian elation group). These works will only have a small overlap with the present notes.

I describe, in detail, the beautiful result of Frohardt which solved Kantor's conjecture in the case when the number of points of the (elation) quadrangle is at most the number of lines. The latter conjecture is the prime power conjecture for elation quadrangles, and states that the parameters of a finite elation generalized quadrangle are powers of the same prime. Along the same lines of Frohardt's proof, I present a nice proof of X. Chen (which was never published) of a conjecture of Payne on the parameters of skew translation quadrangles (which are elation quadrangles such that any dual i-root involving the elation point is Moufang with respect to the same dual root group). The positivity of this conjecture was independently proven by Dirk Hachenberger (in a more general setting), and his proof is also in these notes.

I will also formulate several new questions, often motivated by obtained results. Once the theory on the standard structural questions is worked out, we concentrate on more specific problems, such as a fundamental question posed by Norbert Knarr on the aforementioned local Moufang conditions (motivated by the idea whether there are other, more natural, definitions for the concept of elation quadrangle).

Another aim is to emphasize the role of special p-groups and Moufang conditions as central aspects of elation quadrangle theory.

In many occasions slightly different proofs are given than those provided in the literature. Also, about seventy exercises of (usually) an elementary character are formulated in the text. Exercises which are somewhat less elementary have been indicated with a superscript "$^{\#}$"; exercises which come with a superscript "^{c}" ought to be even more challenging.

Mental note. Throughout this work, almost always the generalized quadrangles (and related objects) we consider are *finite*, even when this is not explicitly mentioned. When this is not the case, the reader will be able to deduce this.

Finally

The notes presented here are partially based on several lectures I gave on elation quadrangles. In particular, I think of the lecture I presented at the conference "Finite Geometries" in La Roche (2004, Belgium), and several talks at the "Buildings conferences" in Würzburg, Darmstadt and Münster, Germany. Also, I lectured on this subject at the University of Colorado at Denver, USA. These talks were often an inspiration for further research, as were the conversations with members of the audience, such as Bill Kantor, Norbert Knarr, Stanley E. Payne and Markus Stroppel.

A first version of the manuscript was finished during a Research in Pairs stay at the Mathematisches Forschungsinstitut Oberwolfach, together with Stefaan De Winter and Ernie Shult, in April 2007. Revised versions were written during the summer of 2010 and the autumn of 2011. In 2010, the counter example of the conjecture stated in [69] was found.

Finally (really)

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Oberwolfach, April 2007, Ghent, December 2011 Koen Thas