Preface

Large scale geometry is the study of geometric objects viewed from afar. In this type of geometry two objects are considered to be the same, if they look roughly the same from a large distance. For example, when viewed from a greater distance the Earth looks like a point, while the real line is not much different from the space of integers. In the past decades mathematicians have discovered many interesting and beautiful large scale geometric properties, which have important applications in topology, analysis, computer science and data analysis.

Historically, large scale geometric ideas appeared in Mostow's rigidity theorem [172], [173] and its generalization due to Margulis [155], [158], as well as in the work of Švarc [227] and later Milnor [166] and Wolf [243] on growth of groups. The impetus came with Gromov's polynomial growth theorem [105] and since then large scale geometric ideas have entered the world of group theory, where they have become a fundamental tool. By now it is well known that many classical notions are in fact of large scale geometric nature, one example being amenability, defined by von Neumann in his work on the Banach–Tarski paradox in 1929.

One of the motivations for this book is the use of large scale geometric methods in index theory. After the Atiyah–Singer index theorem was proved for compact manifolds, a natural question was whether one can extend such index theorems to non-compact manifolds. Connes and Moscovici developed a higher index theory for covering spaces [62]. Motivated by their work, Roe introduced a higher index theory for general non-compact manifolds [207]. Powerful large scale geometric methods have been introduced to compute higher indices for non-compact manifolds [249], [250]. Such computations have allowed us to obtain significant progress on problems like the Novikov conjecture, the Gromov–Lawson–Rosenberg conjecture, or the zero-in-the-spectrum problem, and this area is in a stage of rapid development. More recently, Guentner, Tessera and Yu [117], [118] have introduced large scale geometric methods to study topological rigidity of manifolds. Their method allows them to prove strong results for the stable Borel conjecture and the bounded Borel conjecture.

At the same time, large scale geometric techniques are also of interest in areas such as Banach space geometry, where the coarse classification of Banach spaces remains an interesting problem. Another area of application is theoretical computer science, where embeddability properties, such as compression of coarse embeddings of discrete metric spaces and graphs, allow one to obtain computational efficiency. Finally, we mention that large scale geometric methods have found interesting applications in large data analysis [52], [236].

Our goal is to provide a gentle and fairly detailed introduction to large scale geometric ideas that are used in the study of index problems. We hope that the text will be accessible to a broad audience, in particular to graduate students and newcomers to the field. We provide detailed proofs for most of the theorems stated in the text and every chapter has several exercises at the end. Additionally, some steps in the proofs are left as exercises to the reader. These omissions are always pointed out in the text.

The book is organized in the following way. In the first chapter we discuss basic properties of the coarse category, the geometric viewpoint on finitely generated groups, and we conclude with a section on Gromov hyperbolicity. In the second chapter we give a detailed overview of the notions of asymptotic dimension and decomposition complexity. The third chapter covers amenability of groups. It is slightly shorter, since amenability, as a classical notion, is already a subject of several excellent monographs. In the fourth chapter we discuss the notion of property A for metric spaces and its connections to amenability. In the fifth chapter, coarse embeddings into Banach spaces are studied. The sixth chapter is about affine isometric actions of groups on Banach spaces, in particular a-T-menability (also known as the Haagerup property), Kazhdan's property (T) and constructions of expanders. In the last chapter we introduce elements of large scale algebraic topology: uniformly finite homology and coarse homology theories. The appendix provides a brief survey of applications of coarse geometric properties, discussed in the text, to higher index theory of elliptic operators, and to topological and geometric rigidity.

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