

# Preface

Our original purpose in writing this book was to provide a brief manual, perhaps more aptly called a guide book, that would cover the contents of a basic one-semester course in complex analysis as described in most university curricula. The result, however, has been a more extended text that does not fit into a semester course but is rather appropriate for a variety of advanced courses. It also contains some material that is not usually found in the textbook literature of complex variables. For this reason we hope it will prove to be a good complement to many of the references that are commonly used by both students and teachers.

We wrote this book because we wanted to provide something new, not only in presentation but also in content, when compared with the long and still growing list of complex variable textbooks, many of which have become classics. The starting point was to frame complex analysis within the general framework of mathematical analysis. Although it is possible to present – as many texts do – the complex variable as an isolated branch of study in analysis, we have chosen a different option, namely to seek a maximum number of points of contact with other parts of analysis. This has resulted in the inclusion of some sections that are not common in other texts and a new formulation of some classical results. We highlight a few of them below.

In Chapter 3 we give a real version of the theorems of Cauchy and Cauchy–Goursat. The result is a version of Green’s formula with very weak regularity assumptions, which serves also for classical theorems of vector calculus. In the same chapter, the presentation of Cauchy’s theorem in the context of vector analysis allows us to formulate an approach to the concept of a holomorphic function from a real variable viewpoint, in terms of fields that are simultaneously conservative and solenoidal. The concept of a harmonic function then naturally appears.

Chapter 6 provides a homological version of Green’s formula that can be interpreted as a Green’s formula with multiplicities. With the help of this formula and a standard process of regularization, a question by Ahlfors is answered affirmatively, about the possibility of modifying the proof of Cauchy’s theorem to cover also the case of any locally exact differential form.

Chapter 7 systematically studies harmonic functions and the Laplace operator in the context of real variables in  $\mathbb{R}^n$ , with emphasis on the special case of dimension 2 and the relation with holomorphic functions. The study includes in detail the properties of the Riesz potential of a measure and its importance in solving Poisson’s equation and the Dirichlet and Neumann non-homogeneous problems.

Chapter 9 examines the relationship between Green’s function and conformal mapping, which allows one to prove Riemann’s theorem using the solution of the Dirichlet problem; we also present Koebe’s proof based on the properties of normal

families. The existence of solutions to the Dirichlet problem is proved by Perron's method, which is generalizable to any dimension.

In an analogous way to the Poisson equation, which is the inhomogeneous case of the Laplace equation, Chapter 10 deals with the inhomogeneous Cauchy–Riemann equations. The solution in the general case is obtained using the Runge approximation theorem and is applied to study the Dirichlet problem for the  $\bar{\partial}$  operator.

Chapter 11 is devoted to the study of zero sets of holomorphic functions, and clearly shows the relationship between this topic and the Poisson equation. This allows us to analyze the distribution of zeros of a holomorphic function in terms of their growth.

Finally, the link between real and complex variables also appears in Chapter 12 with the complex Fourier transform or Laplace transform. We provide a proof of the Shannon–Whittaker theorem, well known in information theory, using methods in Chapter 10 on the decomposition of meromorphic functions in simple elements.

To read this text, it is sufficient to have a good knowledge of the topology of the plane and the differential calculus for functions of several real variables. From there, the book is self-contained and gives rigorous proofs of all statements, including a few issues that tend in many books to be treated somewhat superficially. In this regard we emphasize the study, in Chapter 1, of plane domains with regular boundary, including a treatment of the orientation of the border. This study allows us to formulate a precise version of the classic theorems of complex analysis for domains with regular boundary, which are the most used in applications. However, in Chapter 6, we also give the homological version of the fundamental theorems along the line initiated by Ahlfors, which is more general and relates to topological properties of the domain.

The length and structure of the text allows the reader to pursue a variety of paths through it, and to follow a route at different levels. For example, one can follow a basic course in complex variables with Chapters 1 and 2, Chapter 3, Sections 3.1 to 3.5 and Chapters 4, 5 and 8, without the later Sections 8.8 and 8.9. Another possibility is to use, totally or partially, the contents of Chapters 9, 10, 11 and 12 for an expanding course in complex variables.

Given the initial goal of providing maximum interconnection with other parts of analysis, we have put great emphasis on the role of harmonic functions. We have devoted Chapter 7 to them, which is the longest chapter of the book and can be used as an introduction to potential theory. This chapter can be read independently knowing only the content of Chapter 3; on the other hand, Sections 7.7 to 7.12 may require a level of maturity in mathematics a little higher than the preceding chapters.

In general we have devoted much attention to the details of the proofs. However, in some sections of Chapters 7, 10, 11 and 12 the level of precision is lower than for most of the chapters and this can make reading them a little harder.

Each chapter is divided into sections, each section into subsections. All statements (theorems, propositions, lemmas and corollaries), and also examples, are numbered consecutively within each chapter, only observations are numbered separately. The last section of each chapter contains statements of exercises.

Needless to say, in preparing this book we benefited from the work and experience of previous authors. We express our debt to Ahlfors [1], Burckel [3], Gamelin [7], and Saks–Zygmund [11]. We are grateful to Juan Jesús Donaire for his reading of the original, to Lluís Bruna and Miquel Dalmau who read various parts and to Mark Melnikov who provided us with some exercises. They all have made valuable suggestions. We also thank Ignacio Monreal for the translation into English of the Catalan original text.

Finally, the book would not exist without the excellent typographical work of Raquel Hernández, Maria Julià and Rosa Rodríguez. Our thanks to all of them.