Introduction

We learn already in high school that integration plays a central role in mathematics and physics. One encounters integrals in the notions of area or volume, when solving a differential equation, in the fundamental theorem of calculus, in Stokes' theorem, or in classical and quantum mechanics. The first year analysis course at ETH includes an introduction to the Riemann integral, which is satisfactory for many applications. However, it has certain drawbacks, in that some very basic functions are not Riemann integrable, that the pointwise limit of a sequence of Riemann integrable functions need not be Riemann integrable, and that the space of Riemann integrable functions is not complete with respect to the L^1-norm. One purpose of this book is to introduce the *Lebesgue integral,* which does not suffer from these drawbacks and agrees with the Riemann integral whenever the latter is defined. Chapter 1 introduces abstract integration theory for functions on measure spaces. It includes proofs of the Lebesgue monotone convergence theorem, the Fatou lemma, and the Lebesgue dominated convergence theorem. In Chapter 2 we move on to outer measures and introduce the Lebesgue measure on Euclidean space. Borel measures on locally compact Hausdorff spaces are the subject of Chapter 3. Here the central result is the Riesz representation theorem. In Chapter 4 we encounter L^p spaces and show that the compactly supported continuous functions form a dense subspace of L^p for a regular Borel measure on a locally compact Hausdorff space when $p < \infty$. Chapter 5 is devoted to the proof of the Radon–Nikodým theorem about absolutely continuous measures and to the proof that L^q is naturally isomorphic to the dual space of L^p when $1/p + 1/q = 1$ and $1 < p < \infty$. Chapter 6 deals with differentiation. Chapter 7 introduces product measures and contains a proof of Fubini's Theorem, an introduction to the convolution product on $L^1(\mathbb{R}^n)$, and a proof of the Calderón–Zygmund inequality. Chapter 8 constructs Haar measures on locally compact Hausdorff groups.

Despite the overlap with the book of Rudin [17], there are some differences in exposition and content. A small expository difference is that in Chapter 1 measurable functions are defined in terms of pre-images of (Borel) measurable sets rather than pre-images of open sets. The Lebesgue measure in Chapter 2 is introduced in terms of the Lebesgue outer measure instead of as a corollary of the Riesz representation theorem. The notion of a Radon measure on a locally compact Hausdorff space in Chapter 3 is defined in terms of inner regularity, rather than outer regularity together with inner regularity on open sets. This leads to a somewhat different formulation of the Riesz representation theorem (which includes the result as formulated by Rudin). In Chapters 4 and 5 it is shown that $L^q(\mu)$ is isomorphic to the dual space of $L^p(\mu)$ for all measure spaces (not just the σ-finite ones) whenever $1 < p < \infty$ and $1/p + 1/q = 1$. It is also shown that $L^\infty(\mu)$ is isomorphic to the dual space of $L^1(\mu)$ if and only if the measure space is localizable. Chapter 5 includes a generalized version of the Radon–Nikodým theorem for signed measures, due to Fremlin [4], which does not require that the underlying measure μ is σ-finite. In the formulation of König [8] it asserts that a signed measure admits a μ-density if and only if it is both absolutely continuous and inner regular with respect to μ. In addition, the present book includes a self-contained proof of the Calderón–Zygmund inequality in Chapter 7 and an existence and uniqueness proof for (left and right) Haar measures on locally compact Hausdorff groups in Chapter 8.

The book is intended as a companion for a foundational one-semester lecture course on measure and integration theory and there are many topics that it does not cover. For example, the subject of probability theory is only touched upon briefly at the end of Chapter 1 and the interested reader is referred to the book of Malliavin [13] which covers many additional topics including Fourier analysis, limit theorems in probability theory, Sobolev spaces, and the stochastic calculus of variations. Many other fields of mathematics require the basic notions of measure and integration. They include functional analysis and partial differential equations (see, e.g., Gilbarg–Trudinger [5]), geometric measure theory, geometric group theory, ergodic theory and dynamical systems, and differential topology and geometry.

There are many other textbooks on measure theory that cover most or all of the material in the present book, as well as much more, perhaps from somewhat different view points. They include the book of Bogachev [2] which also contains many historical references, the book of Halmos [6], and the aforementioned books of Fremlin [4], Malliavin [13], and Rudin [17].