Chapter 1
Introduction

1.A Discrete groups as metric spaces

Whenever a group Γ appears in geometry, which typically means that Γ acts on a metric space of some sort (examples include universal covering spaces, Cayley graphs and Rips complexes), the geometry of the space reflects some geometry of the group.

This phenomenon goes back at least to Felix Klein and Henri Poincaré, with tessellations of the half-plane related to subgroups of the modular groups, around 1880. It has then been a well-established tradition to study properties of groups which can be viewed, at least in retrospect, as geometric properties. As a sample, we can mention:

– "Dehn Gruppenbild" (also known as Cayley graphs), used to picture finitely generated groups and their word metrics, in particular knot groups, around 1910. Note that Dehn introduced word metrics for groups in his articles on decision problems (1910–1911).
– Amenability of groups (von Neumann, Tarski, late 20's), and its interpretation in terms of isoperimetric properties (Følner, mid 50's).
– Properties "at infinity", or ends of groups (Freudenthal, early 30's), and structure theorems for groups with two or infinitely many ends (Stallings for finitely generated groups, late 60's, Abels' generalization for totally disconnected locally compact groups, 1974).
– Lattices in Lie groups, and later in algebraic groups over local fields; first a collection of examples, and from the 40's a subject of growing importance, with foundational work by Siegel, Mal'cev, Mostow, L. Auslander, Borel & Harish-Chandra, Weil, Garland, H.C. Wang, Tamagawa, Kazhdan, Raghunathan, Margulis (to quote only them); leading to:
– Rigidity of groups, and in particular of lattices in semisimple groups (Mostow, Margulis, 60's and 70's).
– Growth of groups, introduced independently (!) by A.S. Schwarz (also written Švarc) in 1955 and Milnor in 1968, popularized by the work of Milnor and Wolf, and studied later by Grigorchuk, Gromov, and others, including Guivarc'h, Jenkins, and Losert for locally compact groups.
– Structure of groups acting faithfully on trees (Tits, Bass–Serre theory, Dunwoody decompositions and accessibility of groups, 70's); tree lattices.
– Properties related to random walks (Kesten, late 50's, Guivarc'h, 70's, Varopoulos).
– And the tightly interwoven developments of combinatorial group theory and low dimensional topology, from Dehn to Thurston, and so many others.

From 1980 onwards, for all these reasons and under the guidance of Gromov, in particular of his articles [Grom–81b, Grom–84, Grom–87, Grom–93], the group community has been used to consider a group (with appropriate conditions) as a metric space, and to concentrate on large-scale properties of such metric spaces.

Different classes of groups can be characterized by the existence of metrics with additional properties. We often write **discrete group** for *group*, in view of later sections about *topological groups*, and especially *locally compact groups*. In the discrete setting, we distinguish four classes, each class properly containing the next one:

(all) all discrete groups;
(ct) countable groups;
(fg) finitely generated groups;
(fp) finitely presented groups.

This will serve as a guideline below, in the more general setting of locally compact groups.

Every group Γ has left-invariant metrics which induce the discrete topology, for example that defined by $d(\gamma, \gamma') = 1$ whenever γ, γ' are distinct. The three other classes can be characterized as follows.

Proposition 1.A.1. *Let Γ be a group.*

(ct) *Γ is countable if and only if it has a left-invariant metric with finite balls. Moreover, if d_1, d_2 are two such metrics, the identity map $(\Gamma, d_1) \longrightarrow (\Gamma, d_2)$ is a metric coarse equivalence.*

Assume from now on that Γ is countable.

(fg) *Γ is finitely generated if and only if, for one* (*equivalently, for every*) *metric d as in* (ct)*, the metric space (Γ, d) is coarsely connected. Moreover, a finitely generated group has a left-invariant large-scale geodesic metric with finite balls* (*e.g., a word metric*)*; if d_1, d_2 are two such metrics, the identity map $(\Gamma, d_1) \longrightarrow (\Gamma, d_2)$ is a quasi-isometry.*

(fp) *Γ is finitely presented if and only if, for one* (*equivalently, for every*) *metric d as in* (ct)*, the metric space (Γ, d) is coarsely simply connected.*

The technical terms of the proposition can be defined as follows; we come back to these notions in Sections 3.A, 3.B, and 6.A. A metric space (X, d) is

- **coarsely connected** if there exists a constant $c > 0$ such that, for every pair (x, x') of points of X, there exists a sequence $x_0 = x, x_1, \dots, x_n = x'$ of points in X such that $d(x_{i-1}, x_i) \leq c$ for $i = 1, \dots, n$,
- **large-scale geodesic** if there exist constants $a > 0$, $b \geq 0$ such that the previous condition holds with, moreover, $n \leq ad(x, x') + b$,
- **coarsely simply connected** if every "loop" $x_0, x_1, \dots, x_n = x_0$ of points in X with an appropriate bound on the distances $d(x_{i-1}, x_i)$, can be "deformed by small steps" to a constant loop $x_0, x_0, \dots, x_0$; see 6.A.5 for a precise definition.

If X and Y are metric spaces, a map $f : X \longrightarrow Y$ is

- a **metric coarse equivalence** if,
  - for every $c > 0$, there exists $C > 0$ such that,
    for $x, x' \in X$ with $d_X(x, x') \leq c$, we have $d_Y(f(x), f(x')) \leq C$,

- there exists $g : Y \longrightarrow X$ with the same property, satisfying $\sup_{x \in X} d_X(g(f(x)), x) < \infty$ and $\sup_{y \in Y} d_Y(f(g(y)), y) < \infty$;

– a **quasi-isometry** if there exist $a > 0$, $b \geq 0$ such that
  - $d_Y(f(x), f(x')) \leq a d_X(x, x') + b$ for all $x, x' \in X$,
  - there exists $g : Y \longrightarrow X$ with the same property, satisfying $\sup_{x \in X} d_X(g(f(x)), x) < \infty$ and $\sup_{y \in Y} d_Y(f(g(y)), y) < \infty$.

Two metrics d, d' on a set X are coarsely equivalent [resp. quasi-isometric] if the identity map $(X, d) \longrightarrow (X, d')$ is a metric coarse equivalence [resp. a quasi-isometry].

The characterizations of Proposition 1.A.1 provide conceptual proofs of some basic and well-known facts. Consider for example a countable group Γ, a subgroup of finite index Δ, a finite normal subgroup $N \lhd \Gamma$, and a left-invariant metric d on Γ, with finite balls. Coarse connectedness and coarse simple connectedness are properties invariant by metric coarse equivalence. A straightforward verification shows that the inclusion $\Delta \subset \Gamma$ is a metric coarse equivalence; it follows that Δ is finitely generated (or finitely presented) if and only if Γ has the same property.

It is desirable to have a similar argument for Γ and Γ/N; for this, it is better to rephrase the characterizations (ct), (fg), and (fp) in terms of **pseudo-metrics** rather than in terms of metrics. “Pseudo” means that the pseudo-metric evaluated on two distinct points can be 0.

It is straightforward to adapt to pseudo-metric spaces the technical terms defined above and Proposition 1.A.1.

1.B Discrete groups and locally compact groups

It has long been known that the study of a group Γ can be eased when it sits as a discrete subgroup of some kind in a locally compact group G.

For instance, a cocompact discrete subgroup in a connected locally compact group is finitely generated (Propositions 2.C.3 and 2.C.8). The following two standard examples, beyond the scope of the present book, involve a lattice Γ in a locally compact group G: first, Kazhdan Property (T) is inherited from G to Γ [BeHV–08]; second, if Γ is moreover cocompact in G, cohomological properties of Γ can be deduced from information on G or on its homogeneous spaces [Brow–82, Serr–71].

Other examples of groups Γ that are usefully seen as discrete subgroups of G include finitely generated torsion-free nilpotent groups, which are discrete cocompact subgroups in simply connected nilpotent Lie groups [Ragh–72, Theorem 2.18], and polycyclic groups, in which there are subgroups of finite index that are discrete cocompact subgroups in simply connected solvable Lie groups [Ragh–72, Theorem 4.28]. For some classes of solvable groups, the appropriate ambient group G is not Lie, but a group involving a product of linear groups over non-discrete locally compact fields. For example, for a prime p, the group $\mathbf{Z}[1/p]$ of rational numbers of the form a/p^k (with $a \in \mathbf{Z}$ and $k \geq 0$), and the p-adic field $\mathbf{Q}_p$, we have the diagonal embedding

$$\mathbf{Z}[1/p] \hookrightarrow \mathbf{R} \times \mathbf{Q}_p,$$

the image of which is a discrete subgroup with compact quotient. We refer to Example 8.D.2 for other examples, of the kind

$$\mathbf{Z}[1/6] \rtimes_{1/6} \mathbf{Z} \hookrightarrow (\mathbf{R} \times \mathbf{Q}_2 \times \mathbf{Q}_3) \rtimes_{1/6} \mathbf{Z},$$

the image of which is again discrete with compact quotient.

It thus appears that the natural setting is that of locally compact groups. (This generalization from Lie groups to locally compact groups is more familiar in harmonic analysis than in geometry.) More precisely, for the geometric study of groups, it is appropriate to consider **σ-compact** and **compactly generated locally compact groups**; in the case of discrete groups, σ-compact groups are countable groups, and compact generation reduces to finite generation. Though it is not so well-known, there is a stronger notion of **compact presentation** for locally compact groups, reducing to finite presentation in the case of discrete groups. One of our aims is to expose basic facts involving these properties.

We use **LC-group** as a shorthand for "locally compact group".

1.C Three conditions on LC-groups

Locally compact groups came to light in the first half of XXth century. The notion of compactness slowly emerged from 19th century analysis, and the term was coined by Fréchet in 1906; see [Enge–89, page 136]. Local compactness for spaces was introduced by Pavel Alexandrov in 1923 [Enge–89, page 155]. The first abstract definition of a topological group seems to be that of Schreier [Schr–25]; early contributions to the general theory, from the period 1925–1940, include articles by Leja, van Kampen, Markoff, Freudenthal, Weyl, von Neumann (these are quoted in [Weil–40, chapitre 1]), as well as van Dantzig, Garrett Birkhoff, and Kakutani. Influential books were published by Pontryagin [Pont–39] and Weil [Weil–40], and later by Montgomery & Zippin [MoZi–55].

(Lie groups could be mentioned, but with some care. Indeed, in the early theory of XIXth century, "Lie groups" are local objects, not systematically distinguished from Lie algebras before Weyl, even if some examples are "global", i.e., are topological groups in our sense. See the discussion in [Bore–01], in particular his § I.3.)

Among topological groups, LC-groups play a central role, both in most of what follows and in other contexts, such as ergodic theory and representation theory. Recall that these groups have left-invariant regular Borel measures, as shown by Haar (1933) in the second-countable case, and by Kakutani and Weil (late 30's) in the general case. Conversely, a group with a "Haar measure" is locally compact; see *La réciproque du théorème de Haar*, Appendice I in [Weil–40], and its sharpening in [Mack–57]; see also Appendix B in [GlTW–05]. Gelfand and Raikov (1943) showed that LC-groups have "sufficiently many" irreducible continuous unitary representations [Dixm–69, Corollary 13.6.6]; this does *not* carry over to topological groups (examples of topological groups that are abelian, locally homeomorphic to Banach spaces, and without any non-trivial continuous unitary representations are given in [Bana–83, Bana–91]).

Let G be an LC-group. Denote by G_0 its identity component, which is a normal closed subgroup; G_0 is connected and the quotient group G/G_0 is totally disconnected. Our understanding of connected LC-groups, or more generally of LC-groups G with G/G_0 compact, has significantly increased with the solution of Hilbert Fifth Problem in the early 1950's (Gleason, Montgomery, Zippin, Yamabe, see [MoZi–55]). The seminal work of Willis ([Will–94], see also [Will–01a, Will–01b]) on dynamics of automorphisms of totally disconnected LC-groups allowed further progress. Special attention has been given on normal subgroups and topologically simple totally disconnected LC-groups [Will–07, CaMo–11, CaRW–I, CaRW–II].

The goal of this book is to revisit three finiteness conditions on LC-groups, three natural generalizations of countability, finite generation, and finite presentation.

The first two, σ-compactness and compact generation, are widely recognized as fundamental conditions in various contexts. The third, compact presentation, was introduced and studied by the German school in the 60's (see Section 1.E below), but mainly disappeared from the landscape until recently; we hope to convince the reader of its interest. For an LC-group G, here are these three conditions:

(σC) G is **σ-compact** if it has a countable cover by compact subsets. In analysis, this condition ensures that a Haar measure on G is σ-finite, so that the Fubini theorem is available.

(CG) G is **compactly generated** if it has a compact generating set S.

(CP) G is **compactly presented** if it has a presentation $\langle S \mid R \rangle$ with the generating set S compact in G and the relators in R of bounded length.

Though it does not have the same relevance for the geometry of groups, it is sometimes useful to consider one more condition: G is **second-countable** if its topology has a countable basis; the notion (for LC-spaces) goes back to Hausdorff's 1914 book [Enge–89, page 20].

There is an abundance of groups that satisfy these conditions:

(CP) Compactly presented LC-groups include connected-by-compact groups, abelian and nilpotent compactly generated groups, and reductive algebraic groups over local fields. (Local fields are p-adic fields $\mathbf{Q}_p$ and their finite extensions, and fields $\mathbf{F}_q(\!(t)\!)$ of formal Laurent series over finite fields.)

(CG) Examples of compactly generated groups that are not compactly presented include

  (a) $\mathbf{K}^2 \rtimes \mathrm{SL}_2(\mathbf{K})$, where $\mathbf{K}$ is a local field (Example 8.A.28),

  (b) $(\mathbf{R} \times \mathbf{Q}_2 \times \mathbf{Q}_3) \rtimes_{2/3} \mathbf{Z}$, where the generator of $\mathbf{Z}$ acts by multiplication by $2/3$ on each of $\mathbf{R}$, $\mathbf{Q}_2$, and $\mathbf{Q}_3$ (Example 8.D.2).

(σC) $\mathrm{GL}_n(\mathbf{K})$ and its closed subgroups are second-countable σ-compact LC-groups, for every non-discrete locally compact field $\mathbf{K}$ and every positive integer n.

The condition of σ-compactness rules out uncountable groups with the discrete topology, and more generally LC-groups G with an open subgroup H such that the homogeneous space G/H is uncountable; see Remark 2.A.2(3), Example 2.B.8, Corollaries 2.C.6 & 2.E.7(1), and Example 8.B.7(1). Among σ-compact groups, second countability rules out "very large" compact groups, such as uncountable direct products of non-trivial finite groups.

It is remarkable that each of the above conditions is equivalent to a metric condition, as we now describe more precisely.

1.D Metric characterization of topological properties of LC-groups

A metric on a topological space is **compatible** if it defines the given topology. It is appropriate to relax the condition of compatibility for metrics on topological groups, for at least two reasons.

On the one hand, a σ-compact LC-group G need not be metrizable (as uncountable products of non-trivial finite groups show). However, the Kakutani–Kodaira theorem (Theorem 2.B.6) establishes that there exists a compact normal subgroup K such that G/K has a left-invariant compatible metric $d_{G/K}$; hence $d_G(g, g') := d_{G/K}(gK, g'K)$ defines a natural *pseudo-metric* d_G on G, with respect to which balls are compact. On the other hand, an LC-group G generated by a compact subset S has a **word metric** d_S defined by

$$d_S(g, g') = \min \left\{ n \in \mathbf{N} \;\middle|\; \begin{array}{ll} \text{there exist} & s_1, \ldots, s_n \in S \cup S^{-1} \\ \text{such that} & g^{-1}g' = s_1 \cdots s_n \end{array} \right\}.$$

Note that d_S need not be continuous as a function $G \times G \longrightarrow \mathbf{R}_+$. For example, on the additive group of real numbers, we have $d_{[0,1]}(0, x) = \lceil |x| \rceil := \min\{n \geq 0 \mid n \geq |x|\}$ for all $x \in \mathbf{R}$; hence $1 = d_{[0,1]}(0, 1) \neq \lim_{\varepsilon \to 0} d_{[0,1]}(0, 1 + \varepsilon) = 2$ (with $\varepsilon > 0$ in the limit).

As a consequence, we consider **non-necessarily continuous pseudo-metrics**, rather than compatible metrics. In this context, it is convenient to introduce some terminology.

A pseudo-metric d on a topological space X is **proper** if its balls are relatively compact, and **locally bounded** if every point in X has a neighbourhood of finite diameter with respect to d. A pseudo-metric on a topological group is **adapted** if it is left-invariant, proper, and locally bounded; it is **geodesically adapted** if it is adapted and large-scale geodesic. Basic examples of geodesically adapted metrics are the word metrics with respect to compact generating sets. On an LC-group, a continuous adapted metric is compatible, by Proposition 2.A.9.

Every LC-group has a left-invariant pseudo-metric with respect to which balls of *small enough* radius are compact (Corollary 4.A.4). The classes (σC), (CG) and (CP) can be characterized as follows.

Proposition 1.D.1 (characterization of σ-compact groups). *An* LC-*group G is σ-compact if and only if it has an adapted metric, if and only if it has an adapted pseudo-metric, if and only if it has an adapted continuous pseudo-metric* (*Proposition* 4.A.2).

If G has these properties, two adapted pseudo-metrics on G are coarsely equivalent (*Corollary* 4.A.6).

In particular, a σ-compact group can be seen as a pseudo-metric space, well-defined up to metric coarse equivalence. See also Corollary 3.E.6.

The main ingredients of the proof are standard results: the Birkhoff–Kakutani theorem, which characterizes topological groups that are metrizable, the Struble theorem, which characterizes locally compact groups whose topology can be defined by a proper metric, and the Kakutani–Kodaira theorem, which establishes that σ-compact groups are compact-by-metrizable (Theorems 2.B.2, 2.B.4, and 2.B.6).

Proposition 1.D.2 (characterization of compactly generated groups). *Let G be a σ-compact* LC-*group and d an adapted pseudo-metric on G.*

G is compactly generated if and only if the pseudo-metric space (G, d) is coarsely connected. Moreover, if this is so, there exists a geodesically adapted continuous pseudo-metric on G (*Proposition* 4.B.8).

If G is compactly generated, any two geodesically adapted pseudo-metrics on G are quasi-isometric (*Corollary* 4.B.11).

In particular, word metrics associated to compact generating sets of G are bilipschitz equivalent to each other (*Proposition* 4.B.4).

Alternatively, an LC-group is compactly generated if and only if it has a faithful geometric action (as defined in 4.C.1) on a non-empty geodesic pseudo-metric space (Corollary 4.C.6). In particular a compactly generated group can be seen as a pseudo-metric space, well-defined up to quasi-isometry.

To obtain characterizations with metrics, rather than pseudo-metrics, second-countability is needed:

An LC-*group G is second-countable if and only if it has a left-invariant proper compatible metric* (*Struble Theorem* 2.B.4).

An LC-*group G is second-countable and compactly generated if and only if G has a large-scale geodesic left-invariant proper compatible metric* (*Proposition* 4.B.9).

It is a crucial fact for our exposition that compact presentability can be characterized in terms of adapted pseudo-metrics:

Proposition 1.D.3 (characterization of compactly presented groups). *Let G be a compactly generated* LC-*group and d an adapted pseudo-metric on G.*

G is compactly presented if and only if the pseudo-metric space (G, d) is coarsely simply connected (*Proposition* 8.A.3).

One of the motivations for introducing metric ideas as above appears in the following proposition (see Section 4.C), which extends the discussion at the end of Section 1.A:

Proposition 1.D.4. *Let G be a σ-compact* LC-*group, H a closed subgroup of G such that the quotient space G/H is compact, and K a compact normal subgroup of G. Then the inclusion map $i : H \hookrightarrow G$ and the projection $p : G \twoheadrightarrow G/K$ are metric coarse equivalences with respect to adapted pseudo-metrics on the groups. Moreover, if G is compactly generated and the pseudo-metrics are geodesically adapted, the maps i and p are quasi-isometries.*

In particular, if Γ is a discrete subgroup of G such that G/Γ is compact, G is compactly generated [*respectively compactly presented*] *if and only if Γ is finitely generated* [*resp. finitely presented*].

For other properties that hold (or not) simultaneously for G and Γ, see Remarks 4.A.9 and 4.B.14.

1.E On compact presentations

Despite its modest fame for non-discrete groups, compact presentability has been used as a tool to establish finite presentability of S-arithmetic groups. Consider an algebraic group $\mathbf{G}$ defined over $\mathbf{Q}$. By results of Borel and Harish-Chandra, the group $\mathbf{G}(\mathbf{Z})$ is finitely presented [Bore–62]. Let S be a finite set of primes, and let $\mathbf{Z}_S$ denote the ring of rational numbers with denominators products of primes in S. It is a natural question to ask whether $\mathbf{G}(\mathbf{Z}_S)$ is finitely presented; for classical groups, partial answers were given in [Behr–62].

The group $\mathbf{G}(\mathbf{Z}_S)$ is naturally a discrete subgroup in $\mathcal{G} := \mathbf{G}(\mathbf{R}) \times \prod_{p \in S} \mathbf{G}(\mathbf{Q}_p)$. In [Knes–64], Martin Kneser has introduced the notion of compact presentability, and has shown that $\mathbf{G}(\mathbf{Z}_S)$ is finitely presented if and only if $\mathbf{G}(\mathbf{Q}_p)$ is compactly presented for all $p \in S$; an easy case, that for which $\mathcal{G}/\mathbf{G}(\mathbf{Z}_S)$ is compact, follows from Corollary 8.A.5 below. Building on results of Bruhat and Tits, Behr has then shown that, when $\mathbf{G}$ is reductive, $\mathbf{G}(\mathbf{K})$ is compactly presented for every local field $\mathbf{K}$ [Behr–67, Behr–69]. To sum up, using compact presentability, Kneser and Behr have shown that $\mathbf{G}(\mathbf{Z}_S)$ is finitely presented for every reductive group $\mathbf{G}$ defined over $\mathbf{Q}$ and every finite set S of primes. (Further finiteness conditions for such groups are discussed in several articles by Borel and Serre; we quote [BoSe–76].)

After these articles of Kneser and Behr from the 60's, Abels discussed compact presentation in great detail for solvable linear algebraic groups [Abel–87], showing in particular that several properties of $\mathbf{G}(\mathbf{Z}[1/p])$ are best understood together with those of $\mathbf{G}(\mathbf{Q}_p)$. Otherwise, compact presentations seem to have disappeared from the literature. In his influential article on group cohomology and properties of arithmetic and S-arithmetic groups, Serre does not cite Kneser, and he cites [Behr–62, Behr–69] only very briefly [Serr–71, in particular page 127].

1.F Outline of the book

Chapter 2 contains foundational facts on LC-spaces and groups, the theorems of Birkhoff–Kakutani, Struble, and Kakutani–Kodaira, on metrizable groups, with proofs, and generalities on compactly generated LC-groups. The last section describes results on the structure of LC-groups; they include a theorem from the 30's on compact open subgroups in totally disconnected LC-groups, due to van Dantzig, and (without proofs) results from the early 50's solving the Hilbert Fifth Problem, due among others to Gleason, Montgomery and Zippin, and Yamabe.

Chapter 3 deals with two categories of pseudo-metric spaces that play a major role in our setting: the metric coarse category, in which isomorphisms are closeness classes of metric coarse equivalences (the category well-adapted to σ-compact LC-groups), and the large-scale category, in which isomorphisms are closeness classes of quasi-isometries (the category well-adapted to compactly generated LC-groups). Section 3.C shows how pseudo-metric spaces can be described in terms of their metric lattices, i.e., of their subsets which are both uniformly discrete and cobounded. Section 3.D illustrates these notions by a discussion of notions of growth and amenability in appropriate spaces and groups. Section 3.E, on what we call the coarse category, alludes to a possible variation, involving bornologies rather than pseudo-metrics.

Chapter 4 shows how the metric notions of Chapter 3 apply to LC-groups. In particular, every σ-compact LC-group has an adapted metric (Proposition 1.D.1), and every compactly generated LC-group has a geodesically adapted pseudo-metric (Proposition 1.D.2). Moreover, σ-compact LC-groups are precisely the LC-groups that can act on metric spaces in a "geometric" way, namely by actions that are isometric, cobounded, locally bounded, and metrically proper; and compactly generated LC-groups are precisely the LC-groups that can act geometrically on coarsely connected pseudo-metric spaces (Theorem 4.C.5, sometimes called the fundamental theorem of geometric group theory). Section 4.D illustrates these notions by discussing locally elliptic groups, namely LC-groups in which every compact subset is contained in a compact open subgroup (equivalently LC-groups G with an adapted metric d such that the metric space (G, d) has asymptotic dimension 0).

Chapter 5 contains essentially examples of compactly generated LC-groups, including isometry groups of various spaces.

Chapter 6 deals with the appropriate notion of simple connectedness for pseudo-metric spaces, called coarse simple connectedness. In Section 6.C, we introduce the (2-skeleton of the) Rips complex of a pseudo-metric space.

Chapter 7 introduces bounded presentation, i.e., presentations $\langle S \mid R\rangle$ with arbitrary generating set S and relators in R of bounded length. It is a technical interlude before Chapter 8, on compactly presented groups, i.e., on bounded presentations $\langle S \mid R\rangle$ of topological groups G with S compact in G.

As explained in Chapter 8, an LC-group is compactly presented if and only if the pseudo-metric space (G, d) is coarsely simply connected, for d an adapted pseudo-metric. Important examples of compactly presented LC-groups, include:

(c) Connected-by-compact groups.
(d) Abelian and nilpotent compactly generated LC-groups.
(e) LC-groups of polynomial growth.
(f) Gromov-hyperbolic LC-groups.
(g) $(\mathbf{R} \times \mathbf{Q}_2 \times \mathbf{Q}_3) \rtimes_{1/6} \mathbf{Z}$ (compare with (b) in Section 1.C).
(h) $\mathrm{SL}_n(\mathbf{K})$, for every $n \geq 2$ and every local field $\mathbf{K}$.
(i) Every reductive group over a non-discrete LC-field is compactly presented (this last fact is not proven in this book).

(Items (a) and (b) of the list appear above, near the end of Section 1.C, and refer to LC-groups which are *not* compactly presented.) A large part of Chapter 8 is devoted to the Bieri–Strebel splitting theorem: let G be a compactly presented LC-group such that there exists a continuous surjective homomorphism $\pi : G \twoheadrightarrow \mathbf{Z}$; then G is isomorphic to an HNN-extension $\mathrm{HNN}(H, K, L, \varphi)$ of which the base group H is a compactly generated subgroup of $\ker(\pi)$. Among the prerequisites, there is a section exposing how to extend the elementary theory of HNN-extensions and free products with amalgamation from the usual setting of abstract groups to our setting of LC-groups (Section 8.B).

For another and much shorter presentation of the subject of the book, see [CoHa].

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