Introduction

The study of complex Monge–Ampère equations on compact Kähler manifolds is an important part of the interface between complex analysis and differential geometry. Prescribing the Ricci curvature, constructing Kähler–Einstein metrics, or finding metrics of constant scalar curvature are problems that boil down to (or use in an essential way) solving a complex Monge–Ampère equation. Consider for example

$$(\omega + dd^c\varphi)^n = e^{F(\varphi)}\mu,$$

where ω is a Kähler form, $n = \dim_{\mathbb{C}} X$, $\mu = e^h\omega^n$ is a smooth volume form, and F a smooth function: solving the Calabi conjecture boils down to solving this equation when $F \equiv 0$, while constructing Kähler–Einstein metrics corresponds to $F(\varphi) = -\lambda\varphi$, where λ is a real number whose sign is that of the first Chern class of the underlying manifold.

In local coordinates, this equation for an unknown function φ can be written

$$\det\left(\omega_{\alpha\beta} + \frac{\partial^2\varphi}{\partial z_\alpha \partial \overline{z}_\beta}\right) = e^{F(\varphi)+h} \det\left(\omega_{\alpha\beta}\right),$$

where the Kähler form ω is locally written as

$$\omega = \sum_{\alpha,\beta=1}^{n} \omega_{\alpha\beta}\, i dz_\alpha \wedge d\overline{z}_\beta.$$

It has been realized a few decades ago that it is necessary to deal with mildly singular complex algebraic varieties in order to study the birational classification of smooth complex algebraic manifolds in dimension ≥ 3. It is thus desirable to construct canonical metrics on varieties with mild singularities.

If the singularities are sufficiently mild, one can still make sense of the key objects of study (canonical bundle, Ricci curvature, etc). The search for canonical metrics leads one to study degenerate complex Monge–Ampère equations where ω is merely semi-positive and $\mu = f dV$ is absolutely continuous with respect to Lebesgue measure, with density $f \in L^p$, $p > 1$, that might vanish or be unbounded.

The attempt to solve these degenerate complex Monge–Ampère equations runs into severe analytic difficulties, as the solutions are no longer smooth. One needs to introduce new tools, study weak solutions, and establish partial regularity of the latter. This is the main purpose of this book.

What you will find in this book

The book is divided in four parts and sixteen chapters. The *first part* deals with pluripotential theory in domains of $\mathbb{C}^n$, giving a self-contained presentation of Bedford–Taylor Theory initiated in [BT76, BT82]. This theory allows one to define generalized solutions of complex Monge–Ampère equations and has many applications in complex analysis and dynamics. We haven't tried to address all of these; rather, we merely present in detail those results that we use in the sequel.

In the *second part* we transfer and adapt this theory to the context of compact Kähler manifolds. Since there are no plurisubharmonic functions, quasi-plurisubharmonic functions play the leading role. We also introduce and study finite-energy classes, which play a crucial role in the sequel, following [GZ07] (with complements from [BBEGZ11]). Similar notions were developed by Cegrell in the local setting [Ceg98].

The *third part* is devoted to solving degenerate complex Monge–Ampère equations in various ways. We develop a variational technique in finite-energy classes following [BBGZ13], we explore a viscosity approach [EGZ11], we present a detailed proof of several deep a priori L^∞ estimates that extend Yau's and Kolodziej's cornerstone results [Yau78, Kol98], and we eventually establish the smoothness of solutions to some complex Monge–Ampère equations in Zariski open sets.

We give many applications of these results in the *fourth part*, obtaining Yau's solution to the Calabi conjecture [Yau78] as well as its singular extensions, and showing how to construct (singular) Kähler–Einstein metrics on varieties with mild singularities [EGZ09].

There has been an enormous amount of work on pluripotential theory since Bedford and Taylor laid down the foundations fourty years ago. There have also been many important works on Kähler geometry since Yau's resolution of the Calabi conjecture at about the same time. These fields are vast and developing very rapidly, so it should be made clear that this book is not an attempt to survey these developments. There are good surveys on these, including [Bed93, Kis00, Kol05, PS10, PSS12, G12, Dem13, BEG13].

Our selection of material reflects our own taste and limitations. The heart of the book is Part 3, where we solve various degenerate complex Monge–Ampère equations by using pluripotential tools. We have tried to organize the first two parts in such a way as to quickly reach the point where we can efficiently solve these equations.

There are many interesting subjects that we haven't had the time to cover, notably the recent resolution by Chen, Donaldson, and Sun of the Yau–Tian–Donaldson conjecture in the Kähler–Einstein setting [CDS1, CDS2, CDS3]; it would require at least another book to cover this single result!

Prerequisites

This book is an extended version of lecture notes for a graduate course given by the authors at Université Paul Sabatier (Toulouse, France) in 2011–12. There were sixteen four-hour lectures roughly corresponding to the contents of the chapters.

Of course we could not cover all the material presented in this text. On the average, two thirds of each chapter were covered.

There are plenty of *Exercises* at the end of each chapter. But for a few exceptions, their purpose is either to complete the proof of a result which is only sketched in the text, or to add extra information to the material presented in the chapter. These exercises are not easy and are mostly a pretext to encourage the reader to think further and have complementary readings.

We have succeeded in presenting the major part of this material in a year-long graduate course which corresponds to the fifth year of University studies in France (M2R), but the book is so organized that it should be equally convenient for one (or more) course(s) for PhD students having some interest in one of the following topics

- Complex analysis/geometry.
- Non-linear PDEs of geometric origin.
- Geometric analysis and/or differential geometry.

In the book we briefly recall some fundamental facts from complex analysis and Kähler geometry, but it would certainly be preferable (although not absolutely necessary) for the reader either to have some basic knowledge in, or complement with lectures on

- Complex algebraic geometry [GH, Laz, V].
- Minimal Model Program [Deb01, Kollar].
- Riemannian and complex differential geometry [GHL, T].
- Several complex variables [Dem, Horm90].

We recall several notions from complex algebraic and Kähler manifolds in Chapter 7, together with the various definitions of curvatures and canonical metrics in Kähler geometry. We are aware that this chapter is rather heavy for those who have no background in these fields. Apart from a few striking facts like the dd^c-lemma, they are not used intensively in the sequel, which focuses instead mainly on the study of (degenerate) complex Monge–Ampère equations.

Apart from these exceptions, the first three parts of the book are rather self-contained. The only other tools that we take for granted are Hörmander's solution of $\overline{\partial}$ with L^2-estimates (which we use only on a few occasions) and Schauder's theory for linear elliptic equations of second order (as can be found in any standard PDE textbook, e.g., [GT83]). The fourth part is of a more expository nature.

How are complex Monge–Ampère equations solved?

To solve degenerate complex Monge–Ampère equations of the type, say $(\omega+dd^{c}\varphi)^n = \mu$, one might first try to treat the case when μ is a linear combination of Dirac masses, and then proceed by approximation.

While this approach works quite well for the *real* Monge–Ampère equation [Gut01, Theorem 1.6.2], it fails miserably in the complex case. In general, it's not even clear how to treat the case of a single Dirac mass [CG09]! The difficulty lies in the lack of regularity of quasi-plurisubharmonic functions: while the real Monge–Ampère equation deals with convex functions which are locally Lipschitz, the complex Monge–Ampère equation concerns (quasi-)plurisubharmonic functions, and these are not necessarily continuous (or even bounded). In particular, the complex Monge–Ampère operator is not well defined for all quasi-plurisubharmonic functions, and it is not continuous for the weak topology.

We list below different methods that we develop in the book, starting with the continuity method. It was advocated by Calabi [Cal57] as an approach to solve the Calabi conjecture, and was successfully implemented by Yau [Yau78].

Continuity method The continuity method is a classical tool to try and solve non-linear PDEs. It consists in deforming the PDE of interest into a simpler one for which one already knows the existence of a solution. For the Calabi conjecture, one can use the following path,

$$(\mathrm{CY})_t \qquad (\omega + dd^{c}\varphi_t)^n = \left[te^h + (1-t)\right]\omega^n,$$

where $0 \leq t \leq 1$ and φ_t is a Kähler potential (i.e., $\omega + dd^{c}\varphi_t$ is a Kähler form) normalized so that $\int_X \varphi_t\,\omega^n = 0$ (to guarantee uniqueness). The equation of interest corresponds to $t = 1$, while $(\mathrm{CY})_0$ admits the obvious (and unique) solution $\varphi_0 \equiv 0$.

The goal is then to show that the set $S \subset [0,1]$ of parameters for which there is a (smooth) solution is both open and closed in $[0,1]$: since $[0,1]$ is connected and $0 \in S$, it will then follow that $S = [0,1]$, hence $1 \in S$.

The openness follows by linearizing the equation (this involves the Laplace operator associated to $\omega_t = \omega + dd^{c}\varphi_t$) and using the inverse function theorem. One then needs to establish various a priori estimates to show that S is closed. This step constitutes the core of the proof. We establish these estimates in Chapters 12 and 14 and use them to construct Kähler–Einstein metrics in Chapter 15.

The situation is a bit more delicate when ω is merely semi-positive and big (i.e., with positive volume $\int_X \omega^n > 0$). The Laplace operator Δ_ω is for instance no longer invertible. One can approximate ω by $\omega + \varepsilon\omega_X$, where ω_X is Kähler and $\varepsilon \searrow 0$. When the left-hand side $\mu = f dV$ is moreover degenerate, one can regularize it, use Yau's solution to the Calabi conjecture, and try and pass to the limit. This is the approach used in [EGZ09] to solve the singular Calabi conjecture on mildly singular varieties, as we explain in Chapter 16.

Variational approach For φ a Kähler potential we set

$$\mathcal{F}_\lambda(\varphi) := E(\varphi) + \frac{1}{\lambda}\log\left[\int_X e^{-\lambda\varphi - h}\frac{\omega^n}{V}\right],$$

where $V = \int_X \omega^n = \mathrm{Vol}_\omega(X)$ and

$$E(\varphi) := \frac{1}{(n+1)V}\sum_{j=0}^{n}\int_X \varphi(\omega + dd^c\varphi)^j \wedge \omega^{n-j}$$

is a primitive of the complex Monge–Ampère operator. A function φ is a critical point for $\mathcal{F}_\lambda$ if and only if it satisfies the Euler–Lagrange equation

$$\frac{1}{V}(\omega + dd^c\varphi)^n = \frac{e^{-\lambda\varphi - h}\omega^n}{\int_X e^{-\lambda\varphi - h}\omega^n}.$$

Observe that $\mathcal{F}_\lambda(\varphi + C) = \mathcal{F}_\lambda(\varphi)$, for all $C \in \mathbb{R}$, thus $\mathcal{F}_\lambda$ can be thought of as a functional acting on the metrics $\omega_\varphi := \omega + dd^c\varphi$.

The variational approach consists in trying to extremize the functional $\mathcal{F}_\lambda$ in order to solve the above complex Monge–Ampère equation. This usually requires $\mathcal{F}_\lambda$ to be proper, i.e., $\mathcal{F}(\varphi_j) \to -\infty$ whenever φ_j is a sequence of normalized Kähler potentials with diverging energies $E(\varphi_j) \to -\infty$.

It turns out that $\mathcal{F}_\lambda$ is always proper when $\lambda \le 0$, but might not be proper when $\lambda > 0$. This difference partially explains why it is more difficult to construct Kähler–Einstein metrics of positive curvature.

When X is a Fano manifold with no holomorphic vector fields and $\omega \in c_1(X)$, a theorem of Tian [Tian97] (with complements by Phong, Song, Sturm, and Weinkove [PSSW08]) states that there exists a Kähler–Einstein metric if and only if the functional $\mathcal{F}_1$ is proper.

We will explain a partial generalization of Tian's theorem in Chapter 11 following [BBGZ13]. This generalization has interesting applications in the study of the long-term behavior of the Kähler–Ricci flow.

Viscosity techniques A standard PDE approach to second-order degenerate elliptic equations is the method of viscosity solutions (see [CIL92] for a survey). This method is local in nature, and solves existence and uniqueness problems for weak solutions very efficiently.

Whereas the viscosity theory for real Monge–Ampère equations has been developed by P.L. Lions and others (see e.g., [IL90]), the complex case was not studied until very recently (see [HL09] for a viscosity approach to the Dirichlet problem for the complex Monge–Ampère equation on hyperconvex domains).

A viscosity approach for complex Monge–Ampère equations on compact complex manifolds has been developed in [EGZ11]. Combining pluripotential and viscosity techniques, this approach yields the existence and uniqueness of continuous solutions to complex Monge–Ampère equations of the type

$$(\omega + dd^c \varphi)^n = v,$$

where X is a compact complex manifold in the Fujiki class, v is a semi-positive probability measure with L^p-density, $p > 1$, and $\omega \geq 0$ is a smooth closed semipositive $(1,1)$-form such that $\int_X \omega^n = 1$.

This method gives an alternative proof of Kolodziej's C^0-theorem which does not depend on [Yau78]. It also easily produces the unique negatively curved singular Kähler–Einstein metric in the canonical class of a projective manifold of general type, a result obtained first in [EGZ09].

We explain this viscosity approach first in a local setting in Chapter 6, and then on compact Kähler manifolds in Chapter 13.

Precise Contents

We now describe more precisely the contents of each chapter.

Chapter 1 We start by reviewing the basic properties of harmonic and subharmonic functions in the complex plane $\mathbb{C}$ (mean value, maximum principle, Dirichlet problem in the unit disc, logarithmic potentials).

We then introduce and study the basic properties of plurisubharmonic functions and give many examples and constructions (infinite series, gluing techniques).

We establish important (Montel type) compacity properties. They will play a key role in the sequel, showing that properly normalized families of (quasi-)plurisubharmonic functions are relatively compact for any of the equivalent L^p-topologies, $p \geq 1$.

Chapter 2 We develop in this chapter the basic theory of positive closed currents, as introduced by Lelong [Lel57, Lel67, Lel69].

We first recall a few facts of de Rham theory of currents. The latter can be seen as differential forms with coefficient distributions.

We define positive forms and their duals, positive currents. Fundamental examples of closed positive currents are currents of integration along (the regular part of) analytic subsets, as well as closed and positive differential forms (smooth currents).

We give several equivalent definitions of Lelong numbers of plurisubharmonic functions and present Siu's fundamental analyticity result. We also explain an important minimum principle due to Kiselman, and a (uniform) integrability result of Skoda. We use these results on several occasions in the book.

Chapter 3 In this chapter we develop the first steps of the theory of weak complex Monge–Ampère operators due to Bedford and Taylor [BT76, BT82].

We establish Chern–Levine–Nirenberg type inequalities that allow to define various currents of Monge–Ampère type. These inequalities eventually lead to the definition of the complex Monge–Ampère operator for bounded plurisubharmonic functions.

We show that the complex Monge–Ampère operator is continuous along monotonic sequences, but it is discontinuous for the L^1-topology.

We establish various maximum (comparison and domination) principles. The basic reference here is [BT82], but we have freely used subsequent simplifications by Demailly [Dem91], Cegrell [Ceg88, Ceg04], Blocki [Blo02] and Kolodziej [Kol05]. We propose a simplified approach to the continuity along increasing sequences.

Chapter 4 The main focus in this chapter is the Monge–Ampère capacity, an important concept again due to Bedford and Taylor.

We review the Choquet theory of generalized capacities, which are set functions generalizing measures, in that they do not satisfy the additivity property.

We study relative extremal functions, i.e., Perron type envelopes, which facilitates the (rarely explicit) computation of the Monge–Ampère capacity.

The Monge–Ampère capacity characterizes pluripolar sets: a set has zero outer Monge–Ampère capacity if and only if it is pluripolar (i.e., contained in the $-\infty$ locus of a plurisubharmonic function).

We then show that plurisubharmonic functions are quasi-continuous with respect to the Monge–Ampère capacity (they are continuous outside a set of arbitrarily small capacity).

The standard reference here is [BT82], but we have also benefited from various lecture notes (e.g. [Kol05]).

Chapter 5 We study in Chapter 5 various Dirichlet problems. A landmark reference here is [BT76].

We start by solving the Dirichlet problem for the Laplace equation and use it to obtain a useful characterization of subharmonic functions (the inspiration here comes from viscosity theory, but the latter is only developed in Chapter 6).

We then study the Perron–Bremermann envelope, showing that it is continuous up to the boundary (with a precise control on the modulus of continuity). This envelope is the maximal subsolution to the Dirichlet problem. To show that it is actually a solution, we first treat the case of the unit ball, and then use a classical balayage technique, following [BT76].

We study several additional Dirichlet problems, allowing the right-hand side to depend on the unknown function and treating more general densities. This study depends heavily on stability estimates (works of Cegrell and Kolodziej).

Chapter 6 A standard PDE approach to second-order degenerate elliptic equations is the method of viscosity solutions introduced in [Lio83] (see [CIL92] for a survey).

This method is local in nature and efficiently solves existence and uniqueness problems for weak solutions.

We develop the viscosity approach for complex Monge–Ampère equations on domains of $\mathbb{C}^n$. We first explain the general strategy and then develop the appropriate notions in the complex case. An important difference with the real case is a lack of symmetry between subsolutions and supersolutions.

The key step is to establish the (local) comparison principle for complex Monge–Ampère equations.

Chapter 7 This chapter reviews, without proof, some of the most important concepts and results from complex algebraic and Kähler geometry. We give various definitions and examples (of complex manifolds, tangent bundles, holomorphic vector bundles, sections and metrics, blow-ups, positive currents, Kähler forms, ample line bundle) and say just a few words about Hodge theory.

Among all these important notions, the *canonical bundle* and the $\partial\overline{\partial}$-*lemma* will be used over and over in the sequel. We only say a few words about Hermitian geometry, normal coordinates, canonical connections and curvatures; the reader may wish to complement this chapter with other readings.

The material surveyed here is classical. More information can be found in [GH, Wells, Dem, T, V].

Chapter 8 We establish in Chapter 8 the first basic properties of quasi-plurisubharmonic functions (uniform integrability, compactness, approximation, regularization, extension).

The material presented here uses the local properties of plurisubharmonic functions and Skoda's integrability theorem, which have been studied in the first part, as well as the Ohsawa–Takegoshi L^2-extension theorem, which is used to show that one can weakly approximate a positive (singular) metric of an ample line bundle by (logarithm of modulus of) holomorphic sections of powers of the latter.

This approximation result is not used in the rest of the book, but the Ohsawa–Takegoshi theorem is one of the most important tools in complex analytic geometry. We highly recommend [Bern10, Dem13] for a good survey and [GZhou15] for the most recent developments.

Chapter 9 We introduce in this chapter two families of global intrinsic capacities: the Monge–Ampère and the Alexander–Taylor capacities. As in the local setting, they both characterize pluripolar sets and can be understood through the use of *extremal functions*, envelopes of special subfamilies of quasi-plurisubharmonic functions.

The only prerequisite here is the local theory of Bedford and Taylor. More precisely one needs the following facts studied in the first part of the book:

- continuity, along monotone sequences, of complex Monge–Ampère operators acting on bounded plurisubharmonic functions,
- solution of the homogeneous Dirichlet problem in a ball,

- the maximum principle $\mathbf{1}_{\{v<u\}} MA(\max(u,v)) = \mathbf{1}_{\{v<u\}} MA(u)$,
- pluripolar sets are the zero sets of the Monge–Ampère capacity.

The material presented here essentially comes from [GZ05].

Chapter 10 We extend the definition of the complex Monge–Ampère operator to a class of unbounded quasi-plurisubharmonic functions, the class $\mathcal{E}(X,\omega)$ of ω-psh functions with finite energy.

We define and study various intermediate classes of finite weighted energy, which interpolate between the class of bounded ω-psh functions and those of finite (unweighted) energy.

We show that the class $\mathcal{E}(X,\omega)$ is the largest class of ω-psh functions for which the complex Monge–Ampère operator is well-defined and the maximum principle holds.

The material here is taken from [GZ07], [CGZ08] and [BBGZ13]. An important source of inspiration was the work of Cegrell [Ceg98, Ceg04], who has developed similar objects in domains of $\mathbb{C}^n$.

Chapter 11 Here we start to solve degenerate complex Monge–Ampère equations. We use a variational method, i.e., we try to extremize various functionals whose Euler–Lagrange equations are the complex Monge–Ampère equations we want to solve.

This requires the functionals to have semi-continuity properties (a delicate issue in some cases) and to be *proper*, a property that is possibly lacking on Fano manifolds and explains the non-existence of Kähler–Einstein metrics, as was proved by Tian in [Tian97].

As we build here weak solutions, another delicate issue is to verify that extremizers are critical points. This is done by using an important property of upper envelopes (the projection theorem) whose idea goes back to the fundamental work of Alexandrov [Ale38].

The material of this chapter comes from [BBGZ13], except for the proof of the projection theorem, which is one of the main results of [BeBo10] by Berman and Boucksom. The uniqueness of solutions in its most general form is due to Dinew [Din09].

Chapter 12 We study here when finite energy solutions of complex Monge–Ampère equations are actually locally bounded (or even continuous) off a divisor. We do this by establishing a uniform a priori estimate due to Kolodziej, which generalizes the celebrated a priori estimate of Yau [Yau78].

Refinements of the arguments (which rely on the maximum principle and the comparison of previously introduced capacities) allow one to show that the solutions are continuous and *stable*: if the Monge–Ampère measures converge one to another (in a reasonably strong sense), so do the (normalized) solutions.

The material here is essentially due to Kolodziej [Kol98], with complements from [EGZ09, GZ12] and [DiNL14].

The arguments to show (Hölder) continuity require the reference cohomology class to be Kähler. The generalized capacities introduced by Di Nezza and Lu provide a useful generalization of Kolodziej's technique. It allows us to handle situations where the potentials are not globally bounded and yields a simple proof of the domination principle.

Chapter 13 In this chapter we develop the viscosity approach to the equation

$$(\mathrm{DMA}_v^\varepsilon) \qquad (\omega + dd^c\varphi)^n = e^{\varepsilon\varphi} v,$$

where ω is a closed smooth real $(1,1)$-form on a n-dimensional connected compact complex manifold X, v is a volume form with non-negative continuous density, and $\varepsilon \in \mathbb{R}_+$.

The global comparison principle lies at the heart of the viscosity approach. Once it is established, Perron's method can be applied to produce viscosity solutions. We establish the global comparison principle for $(\mathrm{DMA}_v^\varepsilon)$ and use it to solve the above degenerate complex Monge–Ampère equation.

We eventually show that many solutions of degenerate complex Monge–Ampère equations obtained in previous chapters are continuous on the ample locus of the reference cohomology class. The material here is taken from [EGZ11].

Chapter 14 Here we establish higher-order a priori estimates for the solutions of complex Monge–Ampère equations. We complete here the proof of the existence of smooth solutions to non-degenerate complex Monge–Ampère equations, a celebrated result due to Yau [Yau78].

We also explain a generalization of Yau's result, due to Szekelyhidi and Tosatti [SzTo11] and Di Nezza and Lu [DiNL14], using an elliptic method which requires a refined $\mathcal{C}^2$-estimate due to Păun [Pau08] (the approach of [SzTo11] uses the Kähler–Ricci flow).

The presentation of the material follows closely the ones in [Blo13, BBEGZ11, DiNL14].

Chapter 15 The purpose of this chapter is to introduce and study the canonical metrics of Kähler geometry: extremal, constant scalar curvature, and Kähler–Einstein metrics.

The problem of constructing the latter boils down to solving certain complex Monge–Ampère equations that were studied in the previous chapters. We thus explain how to solve the Calabi Conjecture (following Yau [Yau78]), and how to construct Kähler–Einstein metrics.

We discuss a Riemannian structure on the infinite-dimensional space of Kähler metrics, as introduced by Mabuchi [Mab87] and revisited by Semmes and Donaldson [Sem92, Don99]. We mention without proof the main results of Chen [Che00, CC02] on weak geodesics and discuss how weak geodesics were used by Berndtsson and Berman [Bern15, BerBer14] to show the uniqueness of constant scalar curvature

Kähler metrics, generalizing works of several authors, notably Bando and Mabuchi [BM87] and Donaldson [Don01].

We state the Yau–Tian–Donaldson conjecture and (very) briefly discuss its resolution by Chen, Donaldson, and Sun [CDS1, CDS2, CDS3] in the Kähler–Einstein Fano case.

Chapter 16 The classification of higher-dimensional complex algebraic manifolds (the so called Mori program or MMP=Minimal Model Program) requires one to work on varieties that are "mildly singular". Fortunately, in the presence of such mild singularities, one can still make sense of the canonical bundle, the Ricci curvature, and the Kähler–Einstein equation.

The latter again boils down to solving a complex Monge–Ampère equation which is degenerate, due to the presence of singularities. One can use a log-resolution of singularities, and so work on a smooth manifold, but a price is paid because the reference cohomology class is then merely semi-positive.

In this chapter we present the various notions of positivity for cohomology classes, and many notions of singularities (notably canonical and Kawamata log terminal singularities) that we are going to consider. We then apply the techniques developed in previous chapters to show the existence of singular Kähler–Einstein metrics.

Basic references for this section are [Deb01, Dem13, KM, Laz] and [EGZ09] for the applications to singular Kähler–Einstein metrics.

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Bonne lecture!