Preface

This book originates from a series of lectures given by the author at ETH Zürich during the fall of 2014, in the framework of a Nachdiplomvorlesung, on the Monge–Ampère equation and its applications.

The Monge–Ampère equation is a fully nonlinear, degenerate elliptic equation arising in several problems in the areas of analysis and geometry, such as the prescribed Gaussian curvature equation, affine geometry, and optimal transportation. In its classical form, it consists of prescribing the determinant of the Hessian of a convex function u inside some domain Ω, that is,

$$\det(D^2 u) = f \quad \text{in } \Omega.$$

This is in contrast with the "model" elliptic equation $\Delta v = g$, which prescribes the trace of the Hessian of a function v. There are several boundary conditions that one may consider for u, and in this book we shall focus on Dirichlet boundary conditions prescribing the value of u on $\partial\Omega$.

Our goal is to give a comprehensive introduction to the existence and regularity theory of the Monge–Ampère equation, and to show some selected applications. Although some of the results contained here have already been discussed in the classical book by Gutiérrez [61], recent developments in the theory have motivated us to write a new book on the subject.

In the same spirit as the lectures given at ETH Zürich, the structure of this book follows a "historical" path. More precisely, after a brief introduction in Chapter 1 to the Monge–Ampère equation and its history, Chapter 2 is dedicated to the theory of weak solutions introduced by Alexandrov in the 1940s. This notion of solutions is powerful enough to allow one to obtain existence and uniqueness of weak solutions with any nonnegative Borel measure as a right-hand side. Then in Chapter 3 we address the issue of existence of global smooth solutions. This theory, developed between the 1960s and 1980s, combines the continuity method and some interior a priori estimates due to Pogorelov to show existence of smooth solutions when the domain and the boundary data are smooth. The largest part of this book is devoted to the interior regularity of weak solutions. Specifically, in Chapter 4 we study, in detail, the geometry of solutions, mostly investigated by Caffarelli in the 1990s, and we prove interior $C^{1,\alpha}$, $W^{2,p}$, and $C^{2,\alpha}$ estimates. Finally, in Chapter 5 we describe some extensions and generalizations of the results described in the previous chapters.

Since this theory needs some general knowledge from different areas (linear algebra, convex geometry, measure theory, nonlinear analysis, and PDEs), we have decided to write an appendix where the reader can find precise statements of all the

results that we have used. Whenever possible we have included the proofs of such results, and otherwise we have given a precise reference.

By no means is this book intended to cover all the topics and recent developments in the theory of the Monge–Ampère equation and its variants. Rather, our hope and intent is that, after reading this book, the reader will be able to understand and appreciate contemporary literature on the topic.

The reader may notice that every chapter is divided into several sections and subsections. We believe this will facilitate comprehension and help the reader when moving between different results. Also, for the same reason, long proofs are always split into several steps. Finally, because of the geometric arguments presented, we have included several supporting figures. We hope that the reader will benefit from this presentation style.

This book would not exist without the support and help of many friends and colleagues. First, I have been lucky enough to have Luis Caffarelli as a colleague and department neighbor for many years. His beautiful results on Monge–Ampère have been a constant source of inspiration. Second, I have been fortunate to have Guido De Philippis as a long-time collaborator; investigating the Monge–Ampère equation with him has been tremendously inspiring and enjoyable.

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