

Contents

Preface	v
0 Introduction	1
1 The Laplacian	7
1.1 Laplace and Poisson equations	7
1.2 Mean value property	11
1.3 Quantitative maximum principle	15
2 Poisson equation	21
2.1 Finite measures	21
2.2 Distributional solutions	28
2.3 Superharmonic functions	33
3 Integrable versus measure data	39
3.1 Linear Dirichlet problem	39
3.2 Nonlinear Dirichlet problem	44
4 Variational approach	49
4.1 Sobolev spaces	49
4.2 Minimizers and the Euler–Lagrange equation	62
4.3 Thomas–Fermi energy functional	70
5 Linear regularity theory	77
5.1 Embedding in Sobolev spaces	77
5.2 Weak Lebesgue functions	84
5.3 Critical estimates	87
5.4 Compactness in Sobolev spaces	91
6 Comparison tools	95
6.1 Weak maximum principle	95
6.2 Variants of Kato’s inequality	99
6.3 Localization properties	104
6.4 Inverse maximum principle	106

7	Balayage	111
7.1	Weak normal derivative	111
7.2	Finite charge up to the boundary	117
8	Precise representative	123
8.1	Lebesgue's differentiation theorem	123
8.2	Sobolev functions	128
8.3	Potentials	131
8.4	Kato's inequality revisited	134
9	Maximal inequalities	139
9.1	Integral estimate	139
9.2	Energy estimate	143
9.3	Total charge estimate	150
10	Sobolev and Hausdorff capacities	155
10.1	Comparison properties	155
10.2	Estimating the Sobolev capacity	156
10.3	Estimating the Hausdorff content	162
11	Removable singularities	171
11.1	Schwarz's principle	171
11.2	Polar sets of second order	175
11.3	Carleson's condition	176
12	Obstacle problems	181
12.1	Comparison between capacities	181
12.2	Perron–Remak method	188
12.3	Total charge and energy minimizations	195
13	Families of solutions	205
13.1	Bounded functions	205
13.2	Lebesgue integrable classes	207
13.3	Hölder-continuous solutions	209
14	Strong approximation of measures	215
14.1	Capacitary and density bounds	215
14.2	Radon–Nikodým and Lebesgue decompositions	219
14.3	Perturbation of diffuse measures	225
14.4	Precise density bound	227

15	Traces of Sobolev functions	235
15.1	Existence of the trace	235
15.2	Fractional Sobolev embedding	241
15.3	Range of the trace operator	251
16	Trace inequality	257
16.1	Capacitary, geometric and pointwise interpretations	257
16.2	Hölder continuity revisited	263
16.3	Delocalization of capacitary measures	271
17	Critical embedding	275
17.1	Bounded mean oscillation	275
17.2	Brezis–Merle inequality and beyond	278
17.3	Exponential Sobolev embedding	287
17.4	$W^{1,2}$ and $W^{2,1}$ capacities	291
18	Quasicontinuity	299
18.1	Continuous potentials	299
18.2	Lusin property	303
18.3	Continuity principle	306
19	Nonlinear problems with diffuse measures	311
19.1	Unconditional existence	311
19.2	Measures must be diffuse	315
20	Extremal solutions	321
20.1	Boundary data revisited	321
20.2	Method of sub- and supersolutions	327
20.3	Nonlinear Perron–Remak method	332
21	Absorption problems	337
21.1	Subcritical case	337
21.2	Contraction and stability	341
21.3	Polynomial growth	344
21.4	Exponential growth	347
22	The Schrödinger operator	351
22.1	Strong maximum principle	351
22.2	Existence of solutions with measure data	356
22.3	The bridge	360

Appendices

A	Sobolev capacity	367
A.1	Finite semi-additivity	367
A.2	Outer capacity and pointwise convergence	371
A.3	Strong additivity	376
A.4	Measures on dual Sobolev spaces	381
B	Hausdorff measure	385
B.1	Density estimate	385
B.2	Frostman's lemma	388
B.3	Regularity and uniform approximation	391
C	Solutions and hints to the exercises	395
	Bibliography	415
	Index	449