Chapter 0
Introduction

> "Sans que Gauss s'en soit douté, il a jeté des semences qui sont tombées sur une terre qui n'était pas mûre pour les recevoir. Puis, d'autres ouvriers sont venus qui ont remué ce sol ingrat, l'ont amendé, lui ont apporté les sucs nourriciers nécessaires à sa fécondité, et un jour, après un long sommeil, la graine qui n'était pas morte a germé. La plante qui en est sortie est jeune et vivante et c'est à ses fruits que l'on voit enfin la profondeur de la pensée lointaine d'où elle vient."[1]
>
> Charles de la Vallée Poussin

Classical potential theory originated in the 18th century to study the gravitational potential u generated by some density of mass μ, based on Newton's gravitational theory. In the beginning of the 19th century, it was discovered that the potential satisfies the *Poisson equation* [283]

$$-\Delta u = \mu,$$

given in terms of the *Laplacian*

$$\Delta u = \operatorname{div}(\nabla u) = \operatorname{trace}(D^2 u).$$

During the 1830s, Gauss [145] pursued the electrostatic interpretation of the Poisson equation, where in this case μ is a density of electric charges, positive or not. Even though Gauss's argument is unsatisfactory to nowadays standards, his fundamental ideas would deeply influence PDEs and potential theory in the 20th century. Subsequent works by Dirichlet [201], Riemann [293], and Thompson [323] also relied on minimization arguments involving the energy

$$\int_\Omega |\nabla u|^2,$$

[1] "Without suspecting it, Gauss sowed some seeds that fell on ground that was not ready to receive them. Later, other workers came along who turned this infertile soil, enriched it and gave it the nourishing juices it needed to become fertile; and one day, after a long sleep, the seed which had not died sprouted. The plant that has emerged from it is young and vigorous, and it is by its fruits that we finally see the profundity of that distant thought whence it came."

but those arguments would not survive Weierstrass's criticism, see [339] and [289]. Alternative approaches were then developed by Schwarz [301], Neumann [263], Robin [298] and others to circumvent the lack of rigor around the Dirichlet principle.

Poincaré [280] and [281] gave a major contribution that enables one to solve the linear Dirichlet problem

$$\begin{cases} -\Delta u = \mu & \text{in } \Omega, \\ \quad u = 0 & \text{on } \partial\Omega, \end{cases} \tag{0.1}$$

in the classical sense, using his balayage method. This approach does not rely on variational methods, which were still unavailable by 1890. The core of his idea consists in moving electric charges from inside an open region $\omega \Subset \Omega$ to its boundary $\partial\omega$, without modifying the electrostatic potential outside ω. Poincaré's paper [281], published in the freshly founded American Journal of Mathematics, is also a landmark in the theory of PDEs, where he calls the attention to the three major models of second-order equations: elliptic (Laplace equation), parabolic (heat equation) and hyperbolic (wave equation).

Hilbert tried to rescue the Dirichlet principle by using minimizing sequences of the energy. He suggested that they would have better convergence properties; for instance, convergence up to some subsequence. Hilbert's first attempt [167] to justify the Dirichlet principle based on that idea was sketchy, and the implementation of his program turned out to be harder than expected, even in dimension two [88], [140], [153], [168], [193], [203], and [324]. Monna's book [251] is a good historical source on the development of the Dirichlet principle in the 19$^{\text{th}}$ century.

By introducing the concept of barrier, Lebesgue [195] clarified the fact that the Dirichlet problem should be solved in two steps: first find a solution of the Poisson equation inside the domain, and then verify that the boundary condition is indeed satisfied. Perron [277] and Remak [292] later developed independently an abstract approach that contains Poincaré's balayage method, and implements Lebesgue's strategy as an obstacle problem.

The Perron–Remak method seems to have a different nature compared to variational tools, since we look for the smallest element in a class of superharmonic functions. It is nevertheless a disguised minimization of the total charge

$$\int_\Omega |\Delta u|.$$

We are no longer in a Hilbert space setting, but from this perspective the Perron–Remak method becomes a natural companion to variational obstacle problems (Chapter 12).

Surprising counterexamples by Zaremba [347] and Lebesgue [196] showed that the Dirichlet problem need not always have a continuous solution in $\overline{\Omega}$, due to the possibility of having singular points on the boundary $\partial\Omega$. This was the beginning of modern potential theory, when the attention turned to the full characterization of the singular set of $\partial\Omega$. A major breakthrough to identify singular points and sets was made by Wiener [341] and [342], by introducing the Newtonian capacity associated to the Dirichlet energy.

Generalized solutions of the Poisson equation were also introduced in the 1920s. An important one is given by the Newtonian potential generated by a finite Borel measure μ, see [122]:

$$u(x) = \frac{1}{(N-2)\sigma_N} \int_{\mathbb{R}^N} \frac{d\mu(y)}{|x-y|^{N-2}}. \tag{0.2}$$

This approach was motivated by the fact that the Newtonian potential of a smooth distribution of charges satisfies the Poisson equation (Chapter 1), and that measures naturally describe densities of mass or electric charge, see p. 24 in [197] or Chapter 2 below. The function u is interpreted as the potential generated by the density μ, even though the Poisson equation need no longer be satisfied in a pointwise – classical – sense.

F. Riesz [295] and [296] connected the integral representation (0.2) with the notions of superharmonic and subharmonic functions based on averages (Chapter 2); de la Vallée Poussin [106] developed Poincaré's balayage method in the new setting of densities given by measures (Chapter 7). These advances were being supported by tools from functional analysis and measure theory.

Frostman in his thesis [139] clarified the relation between the capacity, an analytic tool, and a more geometric object, the Hausdorff measure (Chapter 10). Their non-equivalence was further investigated by Carleson [77]. Two important notions in this direction are:

(a) *Sobolev capacities* $\mathrm{cap}_{W^{k,q}}$, that can be used to identify sets which are effectively detected by Sobolev functions (Chapter 8 and Appendix A);

(b) *Hausdorff contents* $\mathcal{H}^s_\infty$, that are better suited to investigate density properties, compared to the usual Hausdorff measures $\mathcal{H}^s$ (Appendix B).

It is remarkable that the two concepts can be interchanged in the formalism of Maz′ya–Adams trace inequalities (Chapter 16). These trace inequalities also provide the equivalence between the $W^{1,1}$ capacity and the $\mathcal{H}^{N-1}_\delta$ Hausdorff capacities that was discovered by Meyers and Ziemer [245], and first suggested by Fleming's pioneer contribution [133].

Frostman was particularly interested in the existence of minimizers of Gauss's energy functional. The solution of Gauss's problem would play an important role in the development of the French school on axiomatic potential theory by Brelot, Cartan, Choquet, Deny, and others, see [46], [47], [48], and [49].

The systematic use of Schwartz's theory of distributions [303] and [304] after World War II provided some solid foundation to weak formulations of linear PDEs, applied until then as an *ad hoc* strategy to solve different specific problems [216]. A notable exception comes from the work of Sobolev, who introduced the concept of distribution of finite order in 1935, which he called *fonctionnelle*, see [311] and [312]; a few years later, he also defined weak derivatives as we use nowadays [313].

Modern PDE methods based on functional analysis, a priori estimates, Sobolev spaces, fixed point, and variational techniques [57] were later applied by Littman, Stampacchia, and Weinberger [213] to tackle the linear Dirichlet problem (0.1) involving measure data. They established in particular the existence and uniqueness of a solution for every finite measure μ (Chapter 3).

Concerning the regularity of such a solution, Stampacchia's truncation method provides zeroth and first-order estimates in the setting of weak Lebesgue (Marcinkiewicz) spaces. Indeed, solutions are weak $L^{\frac{N}{N-2}}$, and their gradients exist and are weak $L^{\frac{N}{N-1}}$ (Chapter 5). We have in particular the existence of a force field $-\nabla u$ associated to any finite measure.

These embeddings should be compared to the classical Sobolev–Gagliardo–Nirenberg inequality, which gives the chain of inclusions:

$$W^{2,1}(\mathbb{R}^N) \subset W^{1,\frac{N}{N-1}}(\mathbb{R}^N) \subset L^{\frac{N}{N-2}}(\mathbb{R}^N).$$

The picture is completed by the Calderón–Zygmund theory on singular integrals, which provides a weak L^1 estimate for the second-order derivative D^2u, see [72] and [149]. This estimate lies at the heart of the more familiar (strong) L^p regularity theory for densities $\mu \in L^p(\Omega)$ such that $1 < p < +\infty$, which is obtained by interpolation.

The companion *nonlinear* Dirichlet problem

$$\begin{cases} -\Delta u + g(u) = \mu & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{0.3}$$

has a two-fold motivation. It arises as an L^1 accretivity condition in the Crandall–Liggett theory applied to the porous medium equation, see Section 10.3 in [334], and is also associated to the Thomas–Fermi theory for densities μ given in terms of finite sums of Dirac masses, each one representing an electric pointwise charge, see [208], [206], and [24]. We assume here that the nonlinearity $g\colon \mathbb{R} \to \mathbb{R}$ is merely a continuous function satisfying the *sign condition*: for every $t \in \mathbb{R}$,

$$g(t)t \geqslant 0.$$

This is called an *absorption* problem, and the nonlinear term in the equation satisfies the contraction property:

$$\int_\Omega |g(u)| \leqslant |\mu|(\Omega).$$

When the density μ belongs to the Lebesgue space $L^2(\Omega)$, this nonlinear Dirichlet problem is variational: the energy functional is bounded from below in the Sobolev space $W_0^{1,2}(\Omega)$, minimizers always exist regardless of the growth of g, and they satisfy the Euler–Lagrange equation (Chapter 4). Solutions also exist when μ merely belongs to $L^1(\Omega)$, see [69] and [144], although the problem is no longer variational. This requires a different argument based on the contraction property and the strong L^1 approximation of the datum (Chapter 3).

The study of the nonlinear Dirichlet problem with measures is more subtle. Bénilan and Brezis [24], see also [50] and [51], discovered in 1975 that for nonlinearities of the form $g(t) = |t|^{p-1}t$ with any exponent $p \geqslant \frac{N}{N-2}$, the nonlinear Dirichlet problem has no solution when μ is a Dirac mass, while for $p < \frac{N}{N-2}$, solutions exist for every finite measure μ.

Bénilan and Brezis's paper [24] on the Thomas–Fermi problem had to wait almost thirty years to be published in its final form, although parts of the manuscript started to circulate by the end of the 1970s, and had a great influence at the time. Since then, the mathematical landscape concerning elliptic PDEs with L^1 and measure data has drastically evolved and new fields have been flourishing:

– equations involving quasilinear or fractional operators;
– nonlinear potential and Calderón–Zygmund theories in Euclidean and metric spaces;
– obstacle problems and regularity of free boundaries;
– removable singularity principles for linear and nonlinear equations;
– measure valued solutions;
– boundary traces and probabilistic interpretation of nonlinear problems.

Concerning the nonlinear Dirichlet Problem (0.3), we are interested in the full characterization of measures for which a solution exists. The answer depends on g, and some techniques involving power ([20] and [24]) and exponential ([333] and [22]) nonlinearities are (Chapter 21):

(*a*) *maximum principles*, adapted to the linear Dirichlet problem, and *Kato's inequality*, to get comparison principles for the nonlinear Dirichlet problem (Chapter 6);

(*b*) *removable singularity principles*, to deduce necessary conditions for the existence of solutions (Chapter 11);

(*c*) *strong approximation of measures*, to handle measures that are merely diffuse with respect to Sobolev capacities or Hausdorff measures (Chapter 14);

(*d*) *trace inequalities*, to have a priori estimates (Chapter 16);

(*e*) *Perron–Remak methods*, based on sub- and supersolutions, to establish the existence of extremal solutions (Chapter 20).

Kato introduced his inequality in a seminal paper [175] on Schrödinger operators $-\Delta + V$ to deal with potentials $V \in L^p$, for any exponent $p \geqslant 1$. His goal was to clarify and avoid unnecessary assumptions usually required by variational tools. Such a program is far from being completed, since many properties of the solutions of the Poisson equation associated to the Schrödinger operator depend on the strength of the singularity of V, see [310]. The Hardy potential $V(x) = c/|x|^2$, for example, is critical in many problems, including the strong maximum principle. We explain in Chapter 22 how the above tools can be successfully adapted to investigate properties of solutions involving singular potentials.

Convention. We systematically denote by Ω an open subset of $\mathbb{R}^N$ in dimension $N \geqslant 1$. Further assumptions on Ω will be explicitly stated when needed. For example, by a *smooth open set* Ω we mean that there exists a smooth (C^∞) function $\phi\colon \mathbb{R}^N \to \mathbb{R}$ such that $\phi < 0$ in Ω, $\phi > 0$ in $\mathbb{R}^N \setminus \overline{\Omega}$, and $\nabla\phi \neq 0$ everywhere on $\partial\Omega$. The *outer normal vector* n is then defined on $\partial\Omega$ by

$$n = \frac{\nabla\phi}{|\nabla\phi|}.$$

We say that a function $u\colon \overline{\Omega} \to \mathbb{R}$ is *smooth* if there exists an infinitely differentiable function $U\colon \mathbb{R}^N \to \mathbb{R}$ such that $U = u$ in $\overline{\Omega}$.