Preface

...I am interested not so much in the human mind as in the marvel of a nature which can obey such an elegant and simple law as the law of gravitation [Newton's law of gravitation]. *Therefore our main concentration will not be on how clever we are to have found it all out, but how clever nature is to pay attention to it.*

(Richard Phillips Feynman, Nobel prize in physics, 1965, [Fey80, p. 14]).

The miracle of the appropriateness of the language of mathematics for the formulation of the laws of physics is a wonderful gift which we neither understand nor deserve. We should be grateful for it and hope that it will remain valid in future research and that it will extend, for better or for worse, to our pleasure even though perhaps also to our bafflement, to wide branches of learning.

(Eugene Paul Wigner, Nobel prize in physics, 1963, [Wig60, p. 14]).

With the example of how mathematics has benefited from and influenced physics, it is clear that if mathematicians do not become involved in biosciences they will simply not be a part of what are likely to be the most important and exciting scientific discoveries of all time.

(James Dickson Murray, [Mur93, Preface, p. v]).

These notes deal with applications of the theory of inhomogeneous function spaces $A^s_{p,q}(\mathbb{R}^n)$ of Besov–Sobolev type where $A \in \{B, F\}$ and $s \in \mathbb{R}$, $0 < p \le \infty$, $0 < q \le \infty$, to distinguished PDE models for hydrodynamics and chemotaxis. Special attention is paid to the classical Keller–Segel equations describing the movement of biological cells in response to chemical gradients.

The aim of Chapter 1 (Preliminaries) is twofold. First we collect some basic notation and a few properties of the functions spaces $A^s_{p,q}(\mathbb{R}^n)$. Secondly we explain what is meant by chemotaxis and how corresponding models may look like. In Chapter 2 we clarify which spaces $A^s_{p,q}(\mathbb{R}^n)$ should be called critical and supercritical with respect to Keller–Segel systems. Mapping properties of related nonlinearities are treated in Chapter 3 in some detail; this may be considered as the prototype of all what follows. The heart of these notes is Chapter 4, where we deal with diverse properties of the classical Keller–Segel equations in $\mathbb{R}^n$ and local in time. This will be complemented in Chapter 5 where we have a closer look at further PDE models for chemotaxis. In Chapter 6 we collect briefly some known related assertions for Navier–Stokes equations preparing a more detailed discussion of chemotaxis Navier–Stokes equations in Chapter 7. Section 5.6 is somewhat outside of the main body of these notes. There we introduce *Faber devices*, subspaces of spaces with dominating mixed smoothness, as reliable numerical schemes to describe peaks, troughs, stripes, spots and other filigree structures as occur in chemotaxis and elsewhere.

Our method is local in time and qualitative as far as the underlying properties of the function spaces involved are concerned. These spaces can also be used for more general equations. This will be indicated occasionally. But we also mention that the peculiar structure of some distinguished equations admits simplifications (shifting the divergence from the nonlinearities to the heat kernels).

We fix our use of $\sim$ (equivalence) as follows. Let I be an arbitrary index set. Then

$$a_i \sim b_i \qquad \text{for } i \in I \text{ (equivalence)} \tag{0.1}$$

for two sets of positive numbers $\{a_i : i \in I\}$ and $\{b_i : i \in I\}$ means that there are two positive numbers c_1 and c_2 such that

$$c_1\, a_i \le b_i \le c_2 a_i \qquad \text{for all } i \in I. \tag{0.2}$$