

Contents

| | |
|--|----|
| Foreword | v |
| 1 Introduction and examples | 1 |
| 1.1 Introduction | 1 |
| 1.2 Some academic examples | 3 |
| 1.2.1 Isoperimetric problems | 3 |
| 1.2.2 Minimal surfaces and capillary surfaces | 5 |
| 1.2.3 Eigenvalue problems | 6 |
| 1.3 Some other examples with applications | 13 |
| 1.3.1 Electromagnetic shaping of liquid metal | 13 |
| 1.3.2 Optimization of a magnet | 15 |
| 1.3.3 Image segmentation | 16 |
| 1.3.4 Identification of cracks or defects | 17 |
| 1.3.5 Reinforcement or insulation problems | 18 |
| 1.3.6 Composite materials and structural optimization | 21 |
| 1.3.7 Examples in aeronautics or fluid mechanics | 22 |
| 2 Topologies on domains of \mathbb{R}^N | 25 |
| 2.1 Why do we need a topology? | 25 |
| 2.2 Different topologies on domains | 26 |
| 2.2.1 Introduction | 26 |
| 2.2.2 Convergence of characteristic functions | 27 |
| 2.2.3 Hausdorff convergence of open sets | 30 |
| 2.2.4 Compact convergence | 38 |
| 2.2.5 Links between the different notions of convergence | 39 |
| 2.2.6 Compactness results | 42 |
| 2.3 Sequence of sets with bounded perimeter | 47 |
| 2.3.1 Definition of the perimeter, properties | 47 |
| 2.3.2 Continuity and compactness | 50 |
| 2.4 Sequences of uniformly regular open sets | 54 |
| 2.5 Exercises | 61 |
| 3 Continuity with respect to domains | 67 |
| 3.1 The Dirichlet problem | 68 |
| 3.1.1 The space H_0^1 and its dual space H^{-1} | 68 |
| 3.1.2 $\text{Lip} \circ H^1 \subset H^1$ | 72 |
| 3.1.3 The Poincaré inequality | 76 |
| 3.1.4 The Dirichlet problem for the Laplacian | 79 |

| | | |
|-------|---|-----|
| 3.2 | Continuity for the Dirichlet problem | 81 |
| 3.2.1 | Problem statement | 81 |
| 3.2.2 | Some easy facts | 81 |
| 3.2.3 | The situation does not depend on f | 84 |
| 3.2.4 | Nondecreasing sequences | 85 |
| 3.2.5 | The dimension-1 case | 86 |
| 3.2.6 | Counterexamples to continuity in dimension 2 | 87 |
| 3.2.7 | Sequence of uniformly Lipschitz open sets | 89 |
| 3.3 | Capacity associated with the H^1 -norm | 92 |
| 3.3.1 | Definitions and first properties | 92 |
| 3.3.2 | Relative capacity and capacitary potential | 96 |
| 3.3.3 | How to compute capacities of sets: Some examples | 101 |
| 3.3.4 | Quasi-continuity and quasi-open sets | 105 |
| 3.3.5 | A new definition of $H_0^1(\Omega)$ | 111 |
| 3.4 | Back to the Dirichlet problem | 113 |
| 3.4.1 | Local perturbation | 114 |
| 3.4.2 | Compact convergence and stable open sets | 115 |
| 3.4.3 | Capacitary-type constraints | 117 |
| 3.5 | The γ -convergence | 120 |
| 3.5.1 | Definition | 120 |
| 3.5.2 | Link with Mosco convergence | 121 |
| 3.5.3 | More operators associated with the $H_0^1\gamma$ -convergence | 123 |
| 3.5.4 | Remarks about nonlinear operators | 125 |
| 3.6 | Quantitative estimates | 125 |
| 3.7 | Continuity for the Neumann problem | 128 |
| 3.7.1 | Introduction | 128 |
| 3.7.2 | A main convergence result | 129 |
| 3.7.3 | More convergence results | 131 |
| 3.7.4 | γ -convergence and the Neumann problem | 133 |
| 3.8 | The bi-Laplacian operator | 136 |
| 3.8.1 | H^2 -capacity | 136 |
| 3.8.2 | Continuity with respect to domains | 138 |
| 3.9 | Exercises | 140 |
| 4 | Existence of optimal shapes | 145 |
| 4.1 | Some geometric problems | 145 |
| 4.1.1 | Isoperimetric problems | 145 |
| 4.1.2 | A generalization | 146 |
| 4.1.3 | Capillary surfaces | 147 |
| 4.2 | Examples of nonexistence | 149 |
| 4.3 | Uniform regularity of admissible shapes | 156 |

| | | |
|-------|--|-----|
| 4.4 | Constraints of capacity type | 158 |
| 4.5 | Minimization of the Dirichlet energy | 159 |
| 4.6 | Effects of perimeter constraints | 166 |
| 4.7 | Monotonicity of the functional | 169 |
| 4.8 | Unbounded class of domains | 178 |
| 4.8.1 | A concentration–compactness argument | 179 |
| 4.8.2 | Notion of shape subsolution | 183 |
| 4.8.3 | A surgery argument | 184 |
| 4.9 | Exercises | 185 |
| 5 | Differentiating with respect to domains | 189 |
| 5.1 | Introduction | 189 |
| 5.2 | Integrals on variable domains | 191 |
| 5.2.1 | Introduction | 191 |
| 5.2.2 | Notation | 192 |
| 5.2.3 | The differentiation formula | 194 |
| 5.2.4 | The proofs | 195 |
| 5.2.5 | Differentiating on intervals and first applications | 199 |
| 5.3 | A model PDE problem | 201 |
| 5.3.1 | Presentation of the problem | 201 |
| 5.3.2 | A formal computation | 201 |
| 5.3.3 | The two main results | 203 |
| 5.3.4 | The proofs | 204 |
| 5.3.5 | Differentiability of higher order | 208 |
| 5.3.6 | Differentiability in regular functional spaces | 209 |
| 5.4 | Integrals on moving boundaries | 211 |
| 5.4.1 | Boundary integrals: Definitions and properties | 212 |
| 5.4.2 | A first statement | 214 |
| 5.4.3 | Some differential geometry | 216 |
| 5.4.4 | Extension of the unit normal vector to a variable domain | 222 |
| 5.4.5 | A general formula for boundary differentiation | 223 |
| 5.5 | Differentiating the Neumann problem | 226 |
| 5.6 | How to differentiate boundary value problems | 230 |
| 5.7 | Differentiation of a simple eigenvalue | 232 |
| 5.8 | Use of the adjoint state | 237 |
| 5.9 | Structure of shape derivatives | 241 |
| 5.9.1 | Introduction and notation | 241 |
| 5.9.2 | A first structure result | 242 |
| 5.9.3 | A selected list of first shape derivatives | 243 |
| 5.9.4 | The structure theorem and its corollaries | 245 |
| 5.9.5 | The proofs | 247 |

| | | |
|-------|--|-----|
| 5.9.6 | A selected list of second shape derivatives | 251 |
| 5.9.7 | Three final remarks | 254 |
| 5.9.8 | Conclusion | 255 |
| 5.10 | Exercises | 256 |
| 6 | Geometric properties of the optimum | 259 |
| 6.1 | Symmetry | 259 |
| 6.1.1 | Introduction | 259 |
| 6.1.2 | The method of Steiner symmetrization | 261 |
| 6.1.3 | Using optimality conditions with a maximum principle | 266 |
| 6.1.4 | Variational methods | 269 |
| 6.1.5 | Using another shape optimization problem | 270 |
| 6.1.6 | A case of nonsymmetry: The Newton problem | 274 |
| 6.2 | Convexity | 277 |
| 6.2.1 | Introduction | 277 |
| 6.2.2 | Comparison with the convex hull | 278 |
| 6.2.3 | Nonnegative curvature | 282 |
| 6.3 | Star-shapedness | 283 |
| 6.3.1 | Use of subsolutions and supersolutions | 284 |
| 6.3.2 | Use of a star-shaped rearrangement | 285 |
| 6.4 | Other geometric properties | 290 |
| 6.4.1 | Connectedness | 290 |
| 6.4.2 | A geometric property of the normal | 294 |
| 6.4.3 | Some other geometric properties | 296 |
| 7 | Relaxation and homogenization | 299 |
| 7.1 | Introduction | 299 |
| 7.1.1 | Γ -convergence. | 299 |
| 7.1.2 | G -convergence | 302 |
| 7.2 | Relaxation for the Dirichlet problem | 305 |
| 7.2.1 | Introduction | 305 |
| 7.2.2 | Completion for γ -convergence | 306 |
| 7.2.3 | Another example | 315 |
| 7.3 | Relaxation by homogenization | 328 |
| 7.3.1 | Presentation of the problem | 328 |
| 7.3.2 | Relaxation | 329 |
| 7.3.3 | Optimality conditions | 332 |
| 7.3.4 | An example of an application | 337 |

[Bibliography](#) 341

[General Index](#) 361

[Index of Mathematicians](#) 367