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Introduction

1.1 Kervaire's six papers on knot theory

Michel Kervaire wrote six papers on knots: [63], [64], [66], [45], [46], and [69].

The first one is both an account of Kervaire's talk at a symposium held at Princeton in spring 1963 in honor of Marston Morse and a report of discussions between several participants in the symposium about Kervaire's results. Reading this paper is really fascinating, since one sees higher-dimensional knot theory emerging. The subject is the fundamental group of knots in higher dimensions.

The second one is the written thesis that Kervaire presented in Paris in June 1964. In fact Kervaire had already obtained a PhD in Zurich under Heinz Hopf in 1955, but in 1964 he applied for a position in France and, at that time, a French thesis was compulsory. In the end, the appointment did not materialize. But the thesis text remains as an article published in the *Bulletin de la S.M.F.* ([64]). As this article is the main reason for writing this text, we name it Kervaire's Paris paper. It is the most important one that Kervaire wrote on knot theory. It can be considered to be the foundational text on knots in higher dimensions, together with contemporary papers by Jerome Levine ([77], [78], and [79]). One should also add to the list the Hirsch–Neuwirth paper [50], which seems to be a development of discussions held during the Morse symposium.

To briefly present the subject of Kervaire's Paris paper, we need a few definitions. A knot $K^n \subset S^{n+2}$ is the image of a differentiable embedding of an n-dimensional homotopy sphere in S^{n+2}. Its exterior $E(K)$ is the complement of an open tubular neighborhood. The exterior has the homology of the circle S^1 by Alexander duality. In short, the subject is the determination of the first homotopy group $\pi_q(E(K))$, which is different from $\pi_q(S^1)$.

Later, Kervaire complained that he had to rush to complete this Paris paper in due time and he had doubts about the quality of its redaction. In fact we find this article to be well written. The exposition is concise and clear, typically in Kervaire's style. The pace is slow in parts that present a difficulty and fast when things are obvious. Now, what was obvious to Kervaire in spring 1964? Among other things, clearly Pontrjagin's construction and surgery. As these techniques are possibly not so well known to a reader fifty years later, we devote Chapters 2 and 3 of this text to a presentation of these matters.

The last chapter of Kervaire's Paris paper is a first attempt at understanding the cobordism of knots in higher dimensions. It contains a complete proof that an evendimensional knot is always cobordant to a trivial knot. For odd-dimensional knots, the subject has a strong algebraic flavor, much related to the isometry groups of quadratic forms and to algebraic number theory. Kervaire liked this algebraic aspect and devoted his third paper [66] to it.

Levine spent the first months of 1977 in Geneva and he gave wonderful lectures on many aspects of knot theory. His presence had a deep influence on several members of the audience, including the two authors of this text! Under his initiative a meeting was organized by Kervaire in Les Plans-sur-Bex in March 1977. The proceedings are recorded in the Springer Lecture Notes volume 685, edited by Jean-Claude Hausmann. The irony of history is that it was precisely at this time that William Thurston made his first announcements, which soon completely shattered classical knot theory and 3-manifolds. See Cameron Gordon's paper [38, p. 44] in the proceedings. The meeting renewed Kervaire's interest in knot theory, and in the following months he wrote his last three papers on the subject.

1.2 A brief description of the content of the book

As promised, we devote Chapters 2 and 3 to some background in differential topology, mainly vector bundles, Pontrjagin's construction, and surgery, culminating with Kervaire–Milnor.

We felt it necessary to devote Chapter 4 to knots in codimension ≥3. Its reading is optional. One reason to do so is that the subject was flourishing at the time, thanks to the efforts of André Haefliger and Levine. It is remarkable that Levine was present in both fields. Another reason is that the two subjects are in sharp contrast. Very roughly speaking one could say that it is a matter of fundamental group. In codimension ≥3 the fundamental group of the exterior is always trivial while it is never so in codimension 2. But more must be said. In codimension ≥3 there are no PL knots, as proved by Christopher Zeeman [137]. Hence everything is a matter of comparison between PL and DIFF. This is the essence of Haefliger's theory of smoothing, written a bit later. On the contrary, in codimension 2 the theories of PL knots and of DIFF knots do not differ much.

In Chapter 5 we present Kervaire's determination of the fundamental group of knots in higher dimensions, together with some of the results from his two papers written with Jean-Claude Hausmann about the commutator subgroup and the center of these groups. Some later developments are also presented.

Chapter 6 exposes Kervaire's results on the first homotopy group $\pi_q(E(K))$, which is different from $\pi_q(S^1)$. For $q \geq 2$ these groups are in fact $\mathbf{Z}[t, t^{-1}]$-modules. Indeed, Kervaire undertakes a first study of such modules, later to be called knot modules by Levine. In our presentation, we include several developments due to Levine.

Up to Kervaire's Paris paper, most of the effort in knot theory went into the construction of knot invariants. They produce necessary conditions for two knots to be equivalent. In Levine's paper [78] a change took place. From Dale Trotter's work it was known that for classical knots, Seifert matrices of equivalent knots are S-equivalent. Levine introduced a class of odd-dimensional knots (he called them simple knots) for which the S-equivalence of the Seifert matrices is both necessary and sufficient for two knots to be isotopic. This is a significant classification result. Indeed, simple knots are already present in Kervaire's Paris paper, but he did not pursue their study that far. Technically, the success of Levine's study is largely due to the fact that these knots bound a very special kind of Seifert hypersurface: a (parallelizable) handlebody. In Chapter 7 we present Levine's work on odd-dimensional simple knots.

Chapter 8 is devoted to higher-dimensional knot cobordism. Levine reduced the determination of these groups to an algebraic problem. A key step in the argument rests on the fact that each knot is cobordant to a simple knot.

In knot theory, the handlebodies one deals with are parallelizable and their boundary is a homotopy sphere. If we keep the parallelizability condition but admit any boundary, the Kervaire–Levine arguments are still valid. This immediately applies to the Milnor fiber of isolated singularities of complex hypersurfaces, as first noticed by Milnor himself and developed by Alan Durfee. Chapter 9 is devoted to that matter. It can be considered a prolongation of Kervaire and Levine's work.

Appendixes A to E are of the kind that can be read independently. They provide basics, comments, and variations on subjects treated elsewhere in our book. In Appendix A we give our conventions on signs, which agree with those in Kauffman–Neumann [57]. This allows us to justify the signs for invariants of some basic algebraic links (examples are given at the end of Chapter 9). Often, authors do not mention their sign conventions and hence one can find other signs in the literature. In Appendix B we prove the existence of Seifert hypersurfaces in a more general context. In Appendixes C and D, we present basics about open book decompositions and parallelizable handlebodies, which are useful in knot theory. Our aim in Appendix E is to present the beautiful result by Mike Hill, Mike Hopkins, and Doug Ravenel about the Kervaire invariant and to explain how this affects the theory of knots in higher dimensions. It is a spectacular way to conclude this book.

A few figures, related to several sections of the book, are available in Appendix F.

1.3 What is a knot?

We use the following definitions.

Definition 1.1. An n-link in S^{n+q} is a compact oriented differential submanifold without boundary $L^n \subset S^{n+q}$. The integer $q \geq 1$ is the codimension of the link. The

n-links L_1^n and L_2^n are equivalent if there exists a diffeomorphism $f : S^{n+q} \longrightarrow S^{n+q}$ such that $f(L_1^n) = L_2^n$, respecting the orientation of S^{n+q} and of the links.

When L^n is a homotopy sphere, we say that $L^n \subset S^{n+q}$ is an n-knot in codimension q.

When L^n is the standard sphere S^n embedded in S^{n+2}, we say that $L^n \subset S^{n+2}$ is a standard n-knot.

The group of an n-link (resp. n-knot), $L^n \subset S^{n+2}$, is the fundamental group of $S^{n+2} \setminus L^n$.

At the beginning of Chapter 4, we present the well-known proof that if two links are equivalent there always exists a diffeomorphism $f : S^{n+q} \longrightarrow S^{n+q}$ that is isotopic to the identity and moves one link to the other. Hence n-links are equivalent if and only if they are isotopic.

In general the boundary of a Milnor fiber is not a homotopy sphere. It is a motivation to explain, in Chapter 7, how Kervaire and Levine's works on simple odd-dimensional knots can be generalized to simple links.

A link $L^n \subset S^{n+2}$ in codimension 2 is always the boundary of a $(n+1)$-dimensional oriented smooth submanifold F^{n+1} in S^{n+2}. We say that F^{n+1} is a **Seifert hypersurface** for $L^n \subset S^{n+2}$ (some authors name it Seifert surface even when $n \geq 2$).

1.4 Knots in the early 1960s

In the early 1960s, knots and links in S^3 (which we call classical knots and links) had been deeply studied. Two objects played a central role in classical knot theory: the group of the knot and the Seifert surfaces. Despite the fact that, at that time, higher-dimensional topology was flourishing, knowledge about higher-dimensional knots and links was very poor. The only interesting examples were obtained by the spinning construction, which goes back to Emil Artin, and by its generalization by Christopher Zeeman, called twist spinning. For example, if G is the group of a classical knot, the spinning produces, for all $n \geq 2$, an n-dimensional knot in S^{n+2} having G as knot group. At that time, specialists knew that an n-link is always the boundary of an oriented, smooth, $(n+1)$-dimensional submanifold of S^{n+2}. Here we call it a Seifert hypersurface of the link. A Seifert hypersurface is always parallelizable.

1.5 Higher-dimensional knots and homotopy spheres

We recall, in Chapter 3, the results of Kervaire and Milnor [67] concerning the group bP^{n+1} of n-dimensional homotopy spheres, which bound parallelizable manifolds.

In particular, bP^{n+1} is trivial when $(n + 1)$ is odd, and is a finite cyclic group when $(n + 1)$ is even. We also recall, in Appendix E (Proposition E.3), that any element of bP^{n+1} can be embedded in S^{n+2}. Conversely an n-dimensional homotopy sphere embedded in S^{n+1} is an element of bP^{n+1} because it is always the boundary of a Seifert hypersurface that is parallelizable. These results lead us to define higher-dimensional knots as embeddings of n-dimensional homotopy spheres in S^{n+1}. Kervaire had another reason to define higher-dimensional knots as embeddings of homotopy spheres. It is easier to construct higher-dimensional knots without having to specify their differentiable structure. When n is even, an n-dimensional knot is always a standard knot because bP^{n+1} is trivial. If $n = 4k - 1$, the signature of the intersection form of a Seifert hypersurface determines the differentiable structure of its boundary. The case $n = 4k + 1$ is treated in Appendix E.

1.6 Links and singularities

The paper [97] by David Mumford marks the beginning of a new era since it puts a light on the role of the topology in studying complex singularities. In Section 9.5 we explain some consequences of the following Mumford result:

Theorem 1.2. *Let Σ be a normal complex surface. If the boundary L of point $P \in \Sigma$ is simply connected, then P is a regular point of Σ.*

Mumford also introduces the concept of "plumbing" to describe the topology of a good resolution of a normal singular point of a complex surface. The reader has to be cautious and should not confuse the plumbings obtained as good resolutions of normal germs of complex hypersurfaces and what we call 2-handlebodies. As explained in Appendix D, following William Browder, there is a way to define $2q$-plumbings as equivalent to q-handlebodies if $q \geq 4$.

The connection between higher-dimensional homotopy spheres and isolated singularities of complex hypersurfaces was established in spring 1966. The story is beautifully (and movingly) told by Egbert Brieskorn in [16, pp. 30–52]. Several mathematicians took part in the events: Egbert Brieskorn, Klaus Jänich, Friedrich Hirzebruch, John Milnor, and John Nash. In June 1966 [15], Egbert Brieskorn proved the following theorem, which is a corollary of his thorough study of the now-named Pham–Brieskorn singularities. The proof rests on the work of Frédéric Pham.

Theorem 1.3. *Let Σ^{2q-1} be a $(2q-1)$-homotopy sphere that bounds a parallelizable manifold. Then there exist $(q + 1)$ integers $a_i \geq 2$, $0 \leq i \leq q$, such that the link $L_f \subset S^{2q+1}$ associated to $f(z_0, \ldots, z_q) = \Sigma_{i=0}^{i=q} z_i^{a_i}$ is a knot diffeomorphic to Σ^{2q-1}.*

With his book *Singular Points of Complex Hypersurfaces* [93], Milnor definitively relates the theory of odd-dimensional links to the study of the embedded topology of a germ $f : (\mathbf{C}^{q+1}, 0) \to (\mathbf{C}, 0)$ with an isolated critical point at the origin in $\mathbf{C}^{q+1}$. In Chapter 9, we recall many important results contained in [93]. In particular we explain how Milnor associates a simple fibered link $L_f^{2q-1} \in S_\epsilon^{2q+1}$ to f. Such a link is called an algebraic link.

In [81], Levine gives, when $q \geq 2$, a classification theorem for simple $(2q-1)$-dimensional knots via the Seifert forms. The knot theory of Kervaire–Levine will directly meet the Milnor theory of algebraic links in a paper by Durfee [29]. Indeed, when $q \geq 3$, Durfee shows that the classification theorem of Levine also gives a classification theorem for algebraic links (always in terms of Seifert forms). Such results are based on the classification of q-handlebodies, which are defined in Appendix D. The classification of algebraic links associated to germs of surfaces in $\mathbf{C}^3$ is still open.

In Section 9.6, we present the notion of joins, following Milnor's paper [88]. Joins appeared to describe the topology of a germ $f = g \oplus h : (\mathbf{C}^{n+m}, 0) \to (\mathbf{C}, 0)$ where $g : (\mathbf{C}^n, 0) \to (\mathbf{C}, 0)$ and $h : (\mathbf{C}^m, 0) \to (\mathbf{C}, 0)$ are germs of holomorphic functions with an isolated critical point at the origin and

$$f(x_1, \ldots, x_n, y_1, \ldots, y_m) = g(x_1, \ldots, x_n) + h(y_1, \ldots, y_m).$$

It gives an inductive method to describe the topology of the Pham–Brieskorn singularities. Kauffman, [56], and Kauffman and Neumann, [57], inspired by the topological behavior of links associated to germs of the type $f = g \oplus h$, constructed topologically big families of higher-dimensional links by induction on the dimension. They have constructions where they control the Seifert form, and others with fibered links where they control the open book decompositions.

1.7 Final remarks

The aim of this book is to pay tribute to Kervaire and to make his work on knots of higher dimensions easier to read for younger generations of mathematicians. Basically it is a mathematical exposition text. Our purpose is not to write a history of knots in higher dimensions. We apologize for not making a list of all papers in the subject. When we present Kervaire's work we try to follow him closely, in order to retain some of the flavor of the original texts. When necessary, we add further contributions often due to Levine. We also propose developments that occurred later. With the passing of time, we find it important to present in detail results on the fundamental group of the knot complement and on simple odd-dimensional knots (and links). Indeed,

(1) the determination of the fundamental group of the knot complement played a key role at the beginning of higher-dimensional knot theory;

(2) higher odd-dimensional simple knots can be classified via their relations with handlebodies; on the one hand this classification induces a classification up to cobordism, and on the other hand, it can be easily generalized to links associated to isolated singular points of complex hypersurfaces.

We have wondered whether to include Levine's name in the title of this text. We have decided not to, although he certainly is the cofounder of higher-dimensional knot theory. But Levine pursued his work much beyond these first years, while Kervaire stopped publishing in the subject (too) early. Hence it would have been difficult to find an equilibrium between them. In fact a study of Levine's work in knot theory should be much longer than this text.

1.8 Conventions and notation

Manifolds and embeddings are $\mathcal{C}^\infty$. Usually, manifolds are compact and oriented. A manifold is **closed** if compact without boundary. The boundary of M is written bM, its interior is $\mathring{M}$, and its closure is $\bar{M}$. Let L be a closed oriented submanifold of a closed oriented manifold M. We denote by $N(L)$ a closed tubular neighborhood of L in M and by $E(L)$ the closure of $M \setminus N(L)$, i.e., $E(L) = M \setminus \mathring{N}(L)$. By definition $E(L)$ is the **exterior** of L in M.

Fibers of vector bundles are vector spaces over the field of real numbers $\mathbf{R}$. The **rank** of a vector bundle over a connected base is the dimension of its fibers.