

# Contents

Introduction . . . . .	xv
<b>0 Preliminaries . . . . .</b>	<b>1</b>
1 Basic Languages . . . . .	2
1.1 Sets and ordered sets . . . . .	2
1.2 Categories . . . . .	3
1.3 Limits . . . . .	4
1.4 Several stabilities for properties of arrows . . . . .	6
Exercises . . . . .	12
2 General topology . . . . .	13
2.1 Some general prerequisites . . . . .	14
2.2 Coherent spaces . . . . .	17
2.3 Valuative spaces . . . . .	29
2.4 Reflexive valuative spaces . . . . .	37
2.5 Locally strongly compact valuative spaces . . . . .	42
2.6 Valuations of locally Hausdorff spaces . . . . .	47
2.7 Some generalities on topoi . . . . .	56
Exercises . . . . .	64
3 Homological algebra . . . . .	67
3.1 Inductive limits . . . . .	68
3.2 Projective limits . . . . .	79
3.3 Coherent rings and modules . . . . .	91
Exercises . . . . .	93
4 Ringed spaces . . . . .	93
4.1 Generalities . . . . .	94
4.2 Sheaves on limit spaces . . . . .	98
4.3 Cohomologies of sheaves on ringed spaces . . . . .	103
4.4 Cohomologies of module sheaves on limit spaces . . . . .	106
Exercises . . . . .	107

5	Schemes and algebraic spaces . . . . .	108
5.1	Schemes . . . . .	109
5.2	Algebraic spaces . . . . .	110
5.3	Derived category calculus . . . . .	111
5.4	Cohomology of quasi-coherent sheaves . . . . .	114
5.5	More basics on algebraic spaces . . . . .	118
	Exercises . . . . .	121
6	Valuation rings . . . . .	121
6.1	Prerequisites . . . . .	122
6.2	Valuation rings and valuations . . . . .	125
6.3	Spectrum of valuation rings . . . . .	128
6.4	Composition and decomposition of valuation rings . . . . .	130
6.5	Center of a valuation and height estimates for Noetherian domains . . . . .	132
6.6	Examples of valuation rings . . . . .	133
6.7	$a$ -adically separated valuation rings . . . . .	134
	Exercises . . . . .	136
7	Topological rings and modules . . . . .	137
7.1	Topology defined by a filtration . . . . .	137
7.2	Adic topology . . . . .	144
7.3	Henselian rings and Zariskian rings . . . . .	154
7.4	Preservation of adicness . . . . .	158
7.5	Rees algebra and $I$ -goodness . . . . .	165
	Exercises . . . . .	167
8	Pairs . . . . .	169
8.1	Pairs . . . . .	170
8.2	Bounded torsion condition and preservation of adicness . . . . .	173
8.3	Pairs and flatness . . . . .	178
8.4	Restricted formal power series ring . . . . .	183
8.5	Adhesive pairs . . . . .	187
8.6	Scheme-theoretic pairs . . . . .	197
8.7	$I$ -valuative rings . . . . .	199
8.8	Pairs and complexes . . . . .	208
	Exercises . . . . .	214

9	Topological algebras of type (V) . . . . .	216
9.1	$a$ -adic completion of valuation rings . . . . .	216
9.2	Topologically finitely generated $V$ -algebras . . . . .	221
9.3	Classical affinoid algebras . . . . .	226
	Exercises . . . . .	231
A	Appendix: Further techniques for topologically of finite type algebras . . . . .	231
A.1	Nagata's idealization trick . . . . .	231
A.2	Standard basis and division algorithm . . . . .	232
	Exercises . . . . .	235
B	Appendix: f-adic rings . . . . .	236
B.1	f-adic rings . . . . .	236
B.2	Modules over f-adic rings . . . . .	242
C	Appendix: Addendum on derived categories . . . . .	244
C.1	Prerequisites on triangulated categories . . . . .	244
C.2	The category of complexes . . . . .	246
C.3	The triangulated category $\mathbf{K}(\mathcal{A})$ . . . . .	249
C.4	The derived category $\mathbf{D}(\mathcal{A})$ . . . . .	251
C.5	Subcategories of $\mathbf{D}(\mathcal{A})$ . . . . .	254
I	<b>Formal geometry</b> . . . . .	257
1	Formal schemes . . . . .	258
1.1	Formal schemes and ideals of definition . . . . .	259
1.2	Fiber products . . . . .	267
1.3	Adic morphisms . . . . .	270
1.4	Formal completion . . . . .	272
1.5	Categories of formal schemes . . . . .	276
1.6	Quasi-compact and quasi-separated morphisms . . . . .	280
1.7	Morphisms of finite type . . . . .	286
	Exercises . . . . .	288
2	Universally rigid-Noetherian formal schemes . . . . .	289
2.1	Universally rigid-Noetherian and universally adhesive formal schemes . . . . .	290
2.2	Morphisms of finite presentation . . . . .	294
2.3	Relation with other notions . . . . .	296
3	Adically quasi-coherent sheaves . . . . .	297
3.1	Complete sheaves and adically quasi-coherent sheaves . . . . .	298
3.2	A.q.c. sheaves on affine formal schemes . . . . .	301

3.3	A.q.c. algebras of finite type . . . . .	306
3.4	A.q.c. sheaves as projective limits . . . . .	307
3.5	A.q.c. sheaves on locally universally rigid-Noetherian formal schemes . . . . .	308
3.6	Complete pull-back of a.q.c. sheaves . . . . .	312
3.7	Admissible ideals . . . . .	314
	Exercises . . . . .	319
4	Several properties of morphisms . . . . .	320
4.1	Affine morphisms . . . . .	320
4.2	Finite morphisms . . . . .	324
4.3	Closed immersions . . . . .	326
4.4	Immersions . . . . .	331
4.5	Surjective, closed, and universally closed morphisms . . . . .	333
4.6	Separated morphisms . . . . .	336
4.7	Proper morphisms . . . . .	341
4.8	Flat and faithfully flat morphisms . . . . .	342
	Exercises . . . . .	350
5	Differential calculus on formal schemes . . . . .	350
5.1	Differential calculus for topological rings . . . . .	351
5.2	Differential invariants on formal schemes . . . . .	356
5.3	Étale and smooth morphisms . . . . .	358
	Exercises . . . . .	365
6	Formal algebraic spaces . . . . .	366
6.1	Adically flat descent . . . . .	367
6.2	Étale topology on adic formal schemes . . . . .	374
6.3	Formal algebraic spaces . . . . .	379
6.4	Several properties of morphisms . . . . .	390
6.5	Universally adhesive and universally rigid-Noetherian formal algebraic spaces .	394
	Exercises . . . . .	395
7	Cohomology theory . . . . .	396
7.1	Cohomologies of a.q.c. sheaves . . . . .	397
7.2	Coherent sheaves . . . . .	398
7.3	Calculus in derived categories . . . . .	399
	Exercises . . . . .	400
8	Finiteness theorem for proper algebraic spaces . . . . .	401
8.1	Finiteness theorem: formulation . . . . .	401
8.2	Generalized Serre's theorem . . . . .	402
8.3	The carving method . . . . .	405

8.4	Proof of Theorem 8.1.3 . . . . .	409
8.5	Application to $I$ -goodness . . . . .	410
	Exercises . . . . .	411
9	GFGA comparison theorem . . . . .	411
9.1	Formulation of the theorem . . . . .	412
9.2	The classical comparison theorem . . . . .	414
9.3	Proof of Theorem 9.1.3 . . . . .	417
9.4	Comparison of Ext modules . . . . .	418
10	GFGA existence theorem . . . . .	419
10.1	Statement of the theorem . . . . .	419
10.2	Proof of Theorem 10.1.2 . . . . .	420
10.3	Applications . . . . .	426
	Exercises . . . . .	427
11	Finiteness theorem and Stein factorization . . . . .	427
11.1	Finiteness theorem for proper morphisms . . . . .	427
11.2	Proof of Theorem 11.1.1 . . . . .	428
11.3	Stein factorization . . . . .	432
A	Appendix: Stein factorization for schemes . . . . .	435
A.1	Pseudo-affine morphisms of schemes . . . . .	435
A.2	Cohomological criterion . . . . .	439
B	Appendix: Zariskian schemes . . . . .	440
B.1	Zariskian schemes . . . . .	440
B.2	Fiber products . . . . .	443
B.3	Ideals of definition and adic morphisms . . . . .	443
B.4	Morphism of finite type and morphism of finite presentation . . . . .	444
C	Appendix: FP-approximated sheaves and GFGA theorems . . . . .	445
C.1	Finiteness up to bounded torsion . . . . .	445
C.2	Global approximation by finitely presented sheaves . . . . .	447
C.3	Finiteness theorem and GFGA theorems . . . . .	451
	Exercises . . . . .	453

<b>II Rigid spaces . . . . .</b>	<b>455</b>
1 Admissible blow-ups . . . . .	457
1.1 Admissible blow-ups . . . . .	457
1.2 Strict transform . . . . .	465
1.3 The cofiltered category of admissible blow-ups . . . . .	469
Exercises . . . . .	470
2 Rigid spaces . . . . .	471
2.1 Coherent rigid spaces and their formal models . . . . .	472
2.2 Admissible topology and general rigid spaces . . . . .	477
2.3 Morphism of finite type . . . . .	485
2.4 Fiber products of rigid spaces . . . . .	486
2.5 Examples of rigid spaces . . . . .	487
Exercises . . . . .	488
3 Visualization . . . . .	488
3.1 Zariski–Riemann spaces . . . . .	490
3.2 Structure sheaves and local rings . . . . .	495
3.3 Points on Zariski–Riemann spaces . . . . .	503
3.4 Comparison of topologies . . . . .	508
3.5 Finiteness conditions and consistency of terminologies . . . . .	510
Exercises . . . . .	513
4 Topological properties . . . . .	514
4.1 Generization and specialization . . . . .	514
4.2 Tubes . . . . .	517
4.3 Separation map and overconvergent sets . . . . .	520
4.4 Locally quasi-compact and paracompact rigid spaces . . . . .	523
Exercises . . . . .	524
5 Coherent sheaves . . . . .	525
5.1 Formal models of sheaves . . . . .	525
5.2 Existence of finitely presented formal models (weak version) . . . . .	531
5.3 Existence of finitely presented formal models (strong version) . . . . .	534
5.4 Integral models . . . . .	537
Exercises . . . . .	539
6 Affinoids . . . . .	540
6.1 Affinoids and affinoid coverings . . . . .	540
6.2 Morphisms between affinoids . . . . .	543
6.3 Coherent sheaves on affinoids . . . . .	547

6.4	Comparison theorem for affinoids . . . . .	548
6.5	Stein affinoids . . . . .	550
6.6	Associated schemes . . . . .	552
	Exercises . . . . .	556
7	Basic properties of morphisms of rigid spaces . . . . .	556
7.1	Quasi-compact and quasi-separated morphisms . . . . .	557
7.2	Finite morphisms . . . . .	558
7.3	Closed immersions . . . . .	561
7.4	Immersions . . . . .	567
7.5	Separated morphisms and proper morphisms . . . . .	568
7.6	Projective morphisms . . . . .	578
	Exercises . . . . .	580
8	Classical points . . . . .	580
8.1	Spectral functors . . . . .	581
8.2	Classical points . . . . .	585
8.3	Noetheriness theorem . . . . .	594
9	GAGA . . . . .	598
9.1	Construction of GAGA functor . . . . .	599
9.2	Affinoid valued points . . . . .	605
9.3	Comparison map and comparison functor . . . . .	608
9.4	GAGA comparison theorem . . . . .	610
9.5	GAGA existence theorem . . . . .	613
9.6	Adic part for non-adic morphisms . . . . .	614
	Exercises . . . . .	618
10	Dimension of rigid spaces . . . . .	618
10.1	Dimension of rigid spaces . . . . .	619
10.2	Codimension . . . . .	629
10.3	Relative dimension . . . . .	629
11	Maximum modulus principle . . . . .	630
11.1	Classification of points . . . . .	630
11.2	Maximum modulus principle . . . . .	635
	Exercises . . . . .	643
A	Appendix: Adic spaces . . . . .	644
A.1	Triples . . . . .	644
A.2	Rigid f-adic rings . . . . .	647
A.3	Adic spaces . . . . .	649

A.4	Rigid geometry and affinoid rings . . . . .	655
A.5	Rigid geometry and adic spaces . . . . .	663
B	Appendix: Tate’s rigid analytic geometry . . . . .	665
B.1	Admissibility . . . . .	665
B.2	Rigid analytic geometry . . . . .	669
C	Appendix: Non-archimedean analytic spaces of Banach type . . . . .	675
C.1	Seminorms and norms . . . . .	675
C.2	Graded valuations . . . . .	677
C.3	Filtered valuations . . . . .	687
C.4	Valuative spectrum of non-archimedean Banach rings . . . . .	699
C.5	Non-archimedean analytic space of Banach type . . . . .	726
C.6	Berkovich analytic geometry . . . . .	738
	Exercises . . . . .	748
D	Appendix: Rigid Zariskian spaces . . . . .	749
D.1	Admissible blow-ups . . . . .	749
D.2	Coherent rigid Zariskian spaces . . . . .	750
E	Appendix: Classical Zariski–Riemann spaces . . . . .	753
E.1	Birational geometry . . . . .	753
E.2	Classical Zariski–Riemann spaces . . . . .	760
F	Appendix: Nagata’s embedding theorem . . . . .	766
F.1	Statement of the theorem . . . . .	766
F.2	Preparation for the proof . . . . .	767
F.3	Proof of Theorem F.1.1 . . . . .	773
F.4	Application: Removing the Noetherian hypothesis . . . . .	775
F.5	Nagata embedding for algebraic spaces . . . . .	776
	Exercises . . . . .	777
	Solutions and hints for exercises . . . . .	779
	Bibliography . . . . .	797
	List of notations . . . . .	811
	Index . . . . .	819