Introduction

In the early stage of its history, *rigid geometry* has been first envisaged in an attempt to construct a *non-Archimedean analytic geometry*, an analogue over non-Archimedean valued fields, such as p-adic fields, of complex analytic geometry. Later, in the course of its development, rigid geometry has acquired several rich structures, considerably richer than being merely 'copies' of complex analytic geometry, which endowed the theory with a great potential of applications. This theory is nowadays recognized by many mathematicians in various research fields to be an important and independent discipline in arithmetic and algebraic geometry. This book is the first volume of our prospective book project, which aims to discuss the rich overall structures of rigid geometry, and to explore its applications.

Before explaining our general perspective on this book project, we first provide an overview of the past developments of the theory.

0. Background. After K. Hensel introduced p-adic numbers by the end of the 19th century, the idea arose of constructing p-adic analogues of already existing mathematical theories that were formerly considered only over the field of real or complex numbers. One such analogue was the theory of complex analytic functions, which had by then already matured into one of the most successful and rich branches of mathematics. Complex analysis saw further developments and innovations later on. Most importantly, from extensive works on complex analytic spaces and analytic sheaves by H. Cartan and J. P. Serre in the mid-20th century, after the pioneering work by K. Oka, arose the new idea that the theory of complex analytic functions should be regarded as part of complex analytic geometry. According to this view, it was only natural to expect the emergence of p-adic analytic geometry, or more generally, non-Archimedean analytic geometry.

However, all first attempts encountered essential difficulties, especially in establishing a reasonable link between the local and global notions of analytic functions. Such a naive approach is, generally speaking, characterized by its inclination to produce a faithful imitation of complex analytic geometry, which can be already seen at the level of point sets and topology of the putative analytic spaces. For example, for the 'complex plane' over $\mathbb{C}_p$ (= the completion of the algebraic closure $\overline{\mathbb{Q}}_p$ of $\mathbb{Q}_p$), one takes the naive point set, that is, $\mathbb{C}_p$ itself, and the topology simply induced by the p-adic metric. Starting from a situation like the one described, one goes on to construct a locally ringed space $X = (X, \mathcal{O}_X)$ by introducing the sheaf $\mathcal{O}_X$ of 'holomorphic functions,' a conventional definition of which is something like this: $\mathcal{O}_X(U)$ for any open subset U is the set of all functions on U that admit a convergent power series expansion at every point. But this leads to an extremely cumbersome situation. Indeed, since the topology of X is totally disconnected, there are too many open subsets, and this causes the patching of functions to be extremely 'wobbly,' so much so that one fails to have good control of the global behavior of analytic functions. For example, if X is the 'p-adic Riemann sphere' $\mathbb{C}_p \cup \{\infty\}$, one would expect that $\mathcal{O}_X(X)$ consists only of constant functions, which, however, is far from being true in this situation.

Let us call the problem described above the *Globalization Problem.*[1] Although in its essence it may be seen, inasmuch as being concerned with patching of analytic functions, as a topological problem, as it will turn out, it deeply links with the issue of how to define the notion of points. In the prehistory of rigid geometry, this Globalization Problem has been one, and perhaps the most crucial one, of the obstacles in the quest for a good non-Archimedean analytic geometry.[2]

1. Tate's rigid analytic geometry. The Globalization Problem found its fundamental solution when J. Tate introduced his *rigid analytic geometry* [94] in a seminar at Harvard University in 1961. Tate's motivation was to justify his construction of the so-called *Tate curves*, a non-Archimedean analogue of 1-dimensional complex tori, build by means of an infinite quotient [95].[3] Tate's solution to the problem consists of the following items:

- a 'reasonable' and 'sufficiently large' class of analytic functions and
- a 'correct' notion of analytic coverings.

Here, one can find behind this idea the influence of A. Grothendieck in at least two ways: first, Tate introduced spaces by means of local characterization in terms of their function rings, as typified by scheme theory; second, he used the machinery of *Grothendieck topology* to define analytic coverings.

Let us briefly review Tate's theory. First of all, Tate introduced the category $\mathbf{Aff}_K$ of so-called *affinoid algebras* over a complete non-Archimedean valuation field K. Each affinoid algebra $\mathfrak{A}$, which is a K-Banach algebra, is considered to be the ring of 'reasonable' analytic functions over the 'space' $\mathrm{Sp}\,\mathfrak{A}$, called the *affinoid*, which is the corresponding object in the dual category $\mathbf{Aff}_K^{\mathrm{opp}}$ of $\mathbf{Aff}_K$. Moreover, based on the notion of *admissible coverings*, he introduced a new 'topology,' in fact, a Grothendieck topology, on $\mathrm{Sp}\,\mathfrak{A}$, which we call the *admissible* topology.

[1] This problem is, in classical literature, usually referred to as the problem in analytic continuation.

[2] In his pioneering works [73] and [74], M. Krasner studied in deep the problem and gave a first general recipe for a meaningful analytic continuation of non-Archimedean analytic functions.

[3] Elliptic curves and elliptic functions over p-adic fields have already been studied by É. Lutz at the suggestion of A. Weil, who was inspired by classical works of Eisenstein (cf. [103], p. 538).

The admissibility imposes, most importantly, a strong finiteness condition on analytic coverings, which establishes close ties between the local and global behaviors of analytic functions, as is well described by the famous *Tate's acyclicity theorem* (**II**.B.2.3). An important consequence of this nice local-to-global connection is the good notion of 'patching' of affinoids, by means of which Tate was able to solve the Globalization Problem, and thus to construct global analytic spaces.

In summary, Tate overcame the difficulty by 'rigidifying' the topology itself by imposing the admissibility condition, a strong restriction on the patching of local analytic functions. It is for this reason that this theory is nowadays called *rigid* analytic geometry.

Aside from the fact that it gave a beautiful solution to the Globalization Problem, it is remarkable that Tate's rigid analytic geometry proved that it is possible to apply Grothendieck's way of constructing geometric objects in the setting of non-Archimedean analytic geometry. Thus, rather than complex analytic geometry, Tate's rigid analytic geometry resembles scheme theory. There seemed to be, however, one technical difference between scheme theory and rigid analytic geometry, which was considered to be quite essential at the time when rigid analytic geometry appeared: rigid analytic geometry had to use Grothendieck topology, not classical point set topology.

There is yet another aspect of rigid analytic geometry reminiscent of algebraic geometry. In order to have a better grasp of the abstractly defined analytic spaces, Tate introduced a notion of points. He defined points of an affinoid $\mathrm{Sp}\,\mathfrak{A}$ to be maximal ideals of the affinoid algebra $\mathfrak{A}$; viz., his affinoids are *visualized* by the maximal spectra, that is, the set of all maximal ideals of affinoid algebras, just like affine varieties in the classical algebraic geometry are visualized by the maximal spectra of finite type algebras over a field. Note that this choice of points is essentially the same as the naive one that we have mentioned before. This notion of points was, despite its naivety, considered to be natural, especially in view of his construction of Tate curves, and practically good enough as far as being concerned with rigid analytic geometry over a fixed non-Archimedean valued field.[4]

2. Functoriality and topological visualization. Tate's rigid analytic geometry has, since its first appearance, proven itself to be useful for many purposes, and been further developed by several authors. For example, H. Grauert and R. Remmert [49] laid the foundations of topological and ring theoretic aspects of affinoid algebras, and R. Kiehl [69] and [70] promoted the theory of coherent sheaves and their cohomologies on rigid analytic spaces.

However, it was widely perceived that rigid analytic geometry still has some essential difficulties, some of which are listed below.

[4] One might be apt to think that Tate's choice of points is an 'easygoing' analogue of the spectra of complex commutative Banach algebras, for which the justification, Gelfand–Mazur theorem, is, however, only valid in complex analytic situation, and actually fails in p-adic situation (see below).

• *Functoriality of points does not hold.* If K'/K is an extension of complete non-Archimedean valuation fields, then one expects to have, for any rigid analytic space X over K, a mapping from the points of the base change $X_{K'}$ to the points of X, which, however, does not exist in general in Tate's framework.

Let us call this problem the *Functoriality Problem*. The problem is linked with the following more fundamental one.

• *The analogue of the Gelfand–Mazur theorem does not hold.* The Gelfand–Mazur theorem states that there exist no Banach field extension of $\mathbb{C}$ other than $\mathbb{C}$ itself. In the non-Archimedean case, in contrast, there exist many Banach K-fields other than finite extensions of K. This would imply that there should be plenty of 'valued points' of an affinoid algebra not factoring through the residue field of a maximal ideal; in other words, there should be much more points than those that Tate has introduced.

It is clear that in order to overcome the difficulties of this kind one has to change the notion of points. More precisely, the problem lies in what to choose as the spectrum of an affinoid algebra. To this, there are at least two solutions:

(I) Gromov–Berkovich style spectrum;

(II) Stone–Zariski style spectrum.

The spectrum of the first style, which turns out to be the 'smallest' spectrum allowing to solve the Functoriality Problem in the category of Banach algebras, consists of height-one valuations, that is, *seminorms* (of a certain type) on affinoid algebras. The resulting point sets carry a natural topology, the so-called Gelfand topology. This kind of spectra was adopted by V. G. Berkovich in his approach to non-Archimedean analytic geometry, so-called *Berkovich analytic geometry* [11]. A nice feature of this approach is that, in principle, it can deal with a wide class of Banach K-algebras, including affinoid algebras, and thus solve the Functoriality Problem (in the category of Banach algebras). Moreover, the spectra of affinoid algebras in this approach are Hausdorff, thereby providing intuitively familiar spaces as the underlying topological spaces of the analytic spaces.

However, the Gelfand topology differs from the admissible topology; it is even weaker, in the sense that, as we will see later, the former topology is a *quotient* of the latter. Therefore, this topology does not solve the Globalization Problem for affinoid algebras compatibly with Tate's solution, and, in order to do analytic geometry, one still has to use the Grothendieck topology just imported from Tate's theory.

It is thus necessary, in order to simultaneously solve the Globalization Problem (for affinoids) and the Functoriality Problem, to further improve the notion of points and the topology. In the second style, the Stone–Zariski style, which we will take up in this book, each spectrum has more points by valuations, not only of height one, but of higher height.[5] It turns out that the topology on the point set thus obtained coincides with the admissible topology on the corresponding affinoid, thus solving the Globalization Problem without using the Grothendieck topology. Moreover, the spectra have plenty of points to solve the Functoriality Problem as well.

As we have seen, to sum up, both the Globalization Problem and the Functoriality Problem are closely linked with the more fundamental issue concerned with the notions of points and topology, that is, the problem of the choice of spectra. What lies behind all this is the philosophical tenet that every notion of space in *commutative* geometry should be accompanied with 'visualization' by means of topological spaces, which we call the *topological visualization* (Figure 1). It can be stated, therefore, that the original difficulties in the early non-Archimedean analytic geometry in general, Globalization and Functoriality, are rooted in the lack of adequate topological visualizations. We will dwell on more on this topic later.

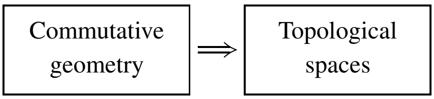

Figure 1. Topological visualization.

3. Raynaud's approach to rigid analytic geometry. To adopt the spectra as in the Stone–Zariski style, in which points are described in terms of valuation rings of arbitrary height, one more or less inevitably has to deal with finer structures, somewhat related to integral structures, of affinoid algebras.[6] The approach is, then, further divided into the following two branches:

(II-a) R. Huber's *adic spaces*[7] [59], [60], and [61];

(II-b) M. Raynaud's viewpoint via formal geometry[8] as a *model geometry* [88].

The last approach, which we will adopt in this book, fits in the general framework in which a geometry as a whole is a package derived from a so-called model geometry. Here is a toy model that exemplifies the framework. Consider, for example, the category of finite-dimensional $\mathbb{Q}_p$-vector spaces. We observe that this

[5] Note that this height tolerance is necessary even for rigid spaces defined over complete valuation fields of height one.

[6] Such a structure, which we call a *rigidification*, will be discussed in detail in **II**, §A.2.(c). In the original Tate rigid analytic geometry, the rigidifications are canonically determined by classical affinoid algebras themselves, and this fact explains why Tate's rigid analytic geometry, unlike the more general Huber's adic geometry, could work without reference to integral models of affinoid algebras.

[7] Note that Huber's theory is based on the choice of integral structures of topological rings. We will give, mainly in **II**, §A, a reasonably detailed account of Huber's theory.

[8] By *formal geometry*, we mean in this book the geometry of formal schemes, developed by A. Grothendieck.

category is equivalent to the quotient category of the category of finitely generated $\mathbb{Z}_p$-modules modulo the Serre subcategory consisting of p-torsion $\mathbb{Z}_p$-modules, since any finite-dimensional $\mathbb{Q}_p$-vector space has a $\mathbb{Z}_p$-lattice, that is, a ‘model’ over $\mathbb{Z}_p$. This suggests that the overall theory of finite-dimensional $\mathbb{Q}_p$-vector spaces is derived from the theory of models, in this case, the theory of finitely generated $\mathbb{Z}_p$-modules.

In our context, what Raynaud discovered on rigid analytic geometry consists of the following statements.

- *Formal geometry,* which has already been established by Grothendieck prior to Tate’s work, can be adopted as a model geometry for Tate’s rigid analytic geometry.
- *The overall theory of rigid analytic geometry arises from Grothendieck’s formal geometry* (Figure 2), which leads to the extremely useful idea that, between formal geometry and Tate’s rigid analytic geometry, one can use theorems in one setting to prove theorems in the other.

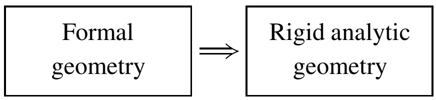

Figure 2. Raynaud’s approach to rigid geometry.

To make more precise the assertion that formal geometry can be a model geometry for rigid analytic geometry, consider, just as in the toy model as above, the category of rigid analytic spaces over K. Raynaud showed that the category of Tate’s rigid analytic spaces (with some finiteness conditions) is equivalent to the quotient category of the category of finite type formal schemes over the valuation ring V of K. Here the ‘quotient’ means inverting all ‘modifications’ (especially, blow-ups) that are ‘isomorphisms over K,’ the so-called *admissible modifications* (*blow-ups*).

There are several important consequences of Raynaud’s discovery; let us mention a few of them. First, guided by the principle that rigid analytic geometry is derived by formal geometry, one can build the theory afresh, starting from *defining* the category of rigid analytic spaces as the quotient category of the category of formal schemes modulo all admissible modifications.[9] Second, Raynaud’s theorem says that *rigid analytic geometry can be seen as the birational geometry of formal schemes*, a novel viewpoint, which motivates one to explore the link with traditional birational geometry. Third, as already mentioned above, the bridge between formal

[9]The rigid spaces obtained in this way are, more precisely, what we call *coherent* (= quasi-compact and quasi-separated) rigid spaces, from which general rigid spaces are constructed by patching.

schemes and rigid analytic spaces, established by Raynaud's viewpoint, gives rise to fruitful interactions between these theories. Especially useful is the fact that theorems in the rigid analytic side can be deduced, at least when one works over complete discrete valuation rings, from theorems in the formal geometry side, available in EGA and SGA works by Grothendieck et al., at least in the Noetherian case.

4. Rigid geometry of formal schemes. We can now describe, along the line of Raynaud's discovery, the basic framework of our rigid geometry that we promote in this book project. For us *rigid geometry is a geometry obtained from a birational geometry of model geometries.* This being so, the main purpose of this book project is to develop such a theory for formal geometry, thus generalizing Tate's rigid analytic geometry and building a more general analytic geometry. Thus to each formal scheme X we associate an object of a resulting category, denoted by X^{rig}, which itself should already be regarded as a rigid space. Then we define general rigid spaces by patching these objects. Note that, here, the rigid spaces are introduced as an 'absolute' object, without reference to a base space.

Among several classes of formal schemes we start with, one of the most important is the class of what we call *locally universally rigid-Noetherian formal schemes*; see **I**.2.1.7. The rigid spaces obtained from this class of formal schemes are called *locally universally Noetherian rigid spaces*, see **II**.2.2.23, which cover most of the analytic spaces that appear in contemporary arithmetic geometry. Note that the formal schemes of the above kind are not themselves locally Noetherian. A technical point resulting from the demand of removing Noetherian hypothesis is that one has to treat *non-Noetherian adic rings* of fairly general kind, for which classical theories, including EGA, do not give us enough tools; for example, valuation rings of arbitrary height are necessary in order to describe points on rigid spaces, and we accordingly need to treat fairly wide class of adic rings over them for describing fibers of finite type morphisms.

Besides, we would like to propose another viewpoint, which classical theory does not offer. Among what Raynaud's theory suggests, the most inspiring is, we think, the idea that rigid geometry should be a birational geometry of formal schemes. We would like to adopt this perspective as one of the core ideas of our theory. In fact, as we will see soon below, it tells us what should be the most natural notion of point of a rigid space, and thus leads to an extremely rich structure concerned with visualizations (that is, spectra), whereby to obtain a satisfactory solution to the above-mentioned Globalization and Functoriality problems. We explain this in the sequel.

5. Revival of Zariski's approach. The birational geometric aspect of our rigid geometry is best explained by means of O. Zariski's classical approach to birational geometry as a model example. Around 1940's, in his attempt to attack the desingularization problem for algebraic varieties, Zariski introduced abstract Riemann spaces for function fields, which we call *Zariski–Riemann spaces*, generalizing the classical valuation-theoretic construction of Riemann surfaces by Dedekind and Weber. This idea has been applied to several other problems in algebraic geometry, including, for example, Nagata's compactification theorem for algebraic varieties.

Let us briefly overview Zariski's idea. Let $Y \hookrightarrow X$ be a closed immersion of schemes (with some finiteness conditions), and set $U = X \setminus Y$. We consider *U-admissible modifications* of X, which are by definition proper birational maps $X' \to X$ that are isomorphisms over U. This class of morphisms contains the subclass consisting of *U-admissible blow-ups*, that is, blow-ups along closed subschemes contained in Y. In fact, U-admissible blow-ups are cofinal in the set of all U-admissible modifications (due to the flattening theorem; cf. **II**, §E.1.(b)). The Zariski–Riemann space, denoted by $\langle X \rangle_U$, is the topological space defined as the projective limit taken along the ordered set of all U-admissible modifications, or equivalently, U-admissible blow-ups, of X. Especially important is the fact that the Zariski–Riemann space $\langle X \rangle_U$ is quasi-compact (essentially due to Zariski [107]; cf. **II**.E.2.5), a fact that is crucial in proving many theorems, for example, the above-mentioned Nagata's theorem.[10]

As is classically known, points of the Zariski–Riemann space $\langle X \rangle_U$ are described in terms of valuation rings. More precisely, these points are in one-to-one correspondence with the set of all morphisms, up to equivalence by 'domination,' of the form Spec $V \to X$, where V is a valuation ring (possibly of height 0), that map the generic point to points in U (see **II**, §E.2.(e) for details). Since the spectra of valuation rings are viewed as 'long paths' (cf. Figure 1 in **0**, §6), one can say intuitively that the space $\langle X \rangle_U$ is like a 'path space' in algebraic geometry (Figure 3).

Now, what we have meant by adopting birational geometry as one of the core ingredients in our theory is that we apply Zariski's approach to birational geometry to the main body of our rigid geometry. Our basic dictionary for doing this, e.g., for rigid geometry over the p-adic field, is as follows:

- $X \longleftrightarrow$ formal scheme of finite type over Spf $\mathbb{Z}_p$;
- $Y \longleftrightarrow$ the closed fiber, that is, the closed subscheme defined by '$p = 0$.'

In this dictionary, the notion of U-admissible blow-ups corresponds precisely to the admissible blow-ups of formal schemes.

6. Birational approach to rigid geometry. As we have already mentioned above, our approach to rigid geometry, called the *birational approach to rigid geometry*, is, so to speak, the combination of Raynaud's algebro-geometric interpretation of

[10] Zariski–Riemann spaces are also used in O. Gabber's unpublished works in 1980's on algebraic geometry problems. Their first appearance in literature in the context of rigid geometry seems to be in [38].

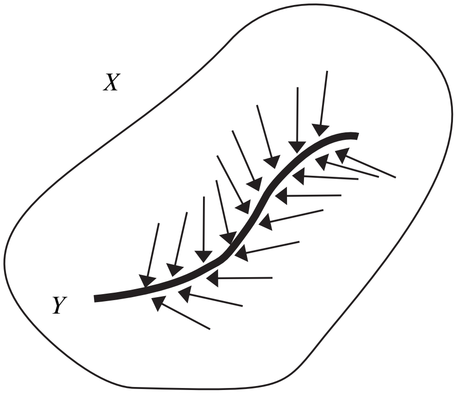

Figure 3. Set-theoretical description of $\langle X \rangle_U$.

rigid analytic geometry, which regards rigid geometry as a birational geometry of formal schemes, and Zariski's classical birational geometry (Figure 4). Most notably, it will turn out that this approach naturally gives rise to the Stone–Zariski style spectrum, which we have already mentioned before.

Raynaud's viewpoint of rigid geometry	+	Zariski's viewpoint of birational geometry

Figure 4. Birational approach to rigid geometry.

A nice point in combining Raynaud's viewpoint and Zariski's viewpoint is that, while the former gives the fundamental recipe for defining rigid spaces, the latter endows them with a 'visualization.' Let us make this more precise, and alongside explain what kind of visualization we attach here to rigid spaces.

As already described earlier, from an adic formal scheme X (of finite ideal type; cf. **I**.1.1.16), we obtain the associated rigid space $\mathscr{X} = X^{\mathrm{rig}}$. Then, suggested by what we have seen in the previous section, we define the *associated Zariski–Riemann space* $\langle \mathscr{X} \rangle$ as the projective limit

$$\langle \mathscr{X} \rangle = \varprojlim X',$$

taken in the category of topological spaces, of all admissible blow-ups $X' \to X$ (Definition **II**.3.2.11). We adopt this space $\langle \mathscr{X} \rangle$ as the topological visualization of the rigid space $\mathscr{X}$. In fact, this space is exactly what we have expected as the topological visualization in the case of Tate's theory, since it can be shown that the canonical topology (the projective limit topology) of $\langle \mathscr{X} \rangle$ actually coincides with the admissible topology.

To explain more about the visualization of rigid spaces, we would like to introduce three kinds of visualizations in a general context. One is the topological visualization, which we have already discussed. The second one, which we name *standard visualization*, is the one that appears in ordinary geometries, as typified by scheme theory; that is, visualization by locally ringed spaces. Recall that an affine scheme, first defined abstractly as an object of the dual category of the category of all commutative rings, can be visualized by a locally ringed space supported on the prime spectrum of the corresponding commutative ring. The third visualization, which we call the *enriched visualization*, or just *visualization* in this book, is given by what we call *triples*:[11] these are objects of the form $(X, \mathcal{O}^+_X, \mathcal{O}_X)$ consisting of a topological space X and two sheaves of topological rings together with an injective ring homomorphism $\mathcal{O}^+_X \hookrightarrow \mathcal{O}_X$ that identifies $\mathcal{O}^+_X$ with an open subsheaf of $\mathcal{O}_X$ such that the pairs $X = (X, \mathcal{O}_X)$ and $X^+ = (X, \mathcal{O}^+_X)$ are locally ringed spaces; in this setting, $\mathcal{O}_X$ is regarded as the structure sheaf of X, while $\mathcal{O}^+_X$ represents the enriched structure, such as an integral structure (whenever it makes sense) of $\mathcal{O}_X$.

The enriched visualization is typified by rigid spaces. The Zariski–Riemann space $\langle \mathscr{X} \rangle$ has two natural structure sheaves, the *integral structure sheaf* $\mathcal{O}^{\mathrm{int}}_{\mathscr{X}}$, defined as the inductive limit of the structure sheaves of all admissible blow-ups of X, and the *rigid structure sheaf* $\mathcal{O}_{\mathscr{X}}$, obtained from $\mathcal{O}^{\mathrm{int}}_{\mathscr{X}}$ by 'inverting the ideal of definition.' What is intended here is that, while the rigid structure sheaf $\mathcal{O}_{\mathscr{X}}$ should, as in Tate's rigid analytic geometry, normally come as the 'genuine' structure sheaf of the rigid space $\mathscr{X}$, the integral structure sheaf $\mathcal{O}^{\mathrm{int}}_{\mathscr{X}}$ represents its integral structure. These data comprise the triple

$$\mathbf{ZR}(\mathscr{X}) = (\langle \mathscr{X} \rangle, \mathcal{O}^{\mathrm{int}}_{\mathscr{X}}, \mathcal{O}_{\mathscr{X}}),$$

called the *associated Zariski–Riemann triple*, which gives the enriched visualization of the rigid space $\mathscr{X}$. That the rigid structure sheaf should be *the* structure sheaf of $\mathscr{X}$ means that the locally ringed space $(\langle \mathscr{X} \rangle, \mathcal{O}_{\mathscr{X}})$ visualizes the rigid space in an ordinary sense, that is, in the sense of standard visualization.

Note that the Zariski–Riemann triple $\mathbf{ZR}(\mathscr{X})$ for a rigid space $\mathscr{X}$ coincides with Huber's adic space associated to $\mathscr{X}$; in fact, the notion of Zariski–Riemann triple gives not only an interpretation of adic spaces, but also a foundation for them via formal geometry, which we establish in this book; see **II**, §A.5 for more details.

Figure 5 illustrates the basic design of our birational approach to rigid geometry, summarizing all what we have discussed so far.

The figure shows a 'commutative' diagram, in which the arrow $(*1)$ is Raynaud's approach to rigid geometry (Figure 2), and the arrow $(*2)$ is the enriched visualization by Zariski–Riemann triples, coming from Zariski's viewpoint. The other visualizations are also indicated in the diagram, the standard visualization

[11] See **II**, §A.1 for the generalities of triples.

by (∗3), and the topological visualization by (∗4); the right-hand vertical arrows represent the respective forgetful functors.

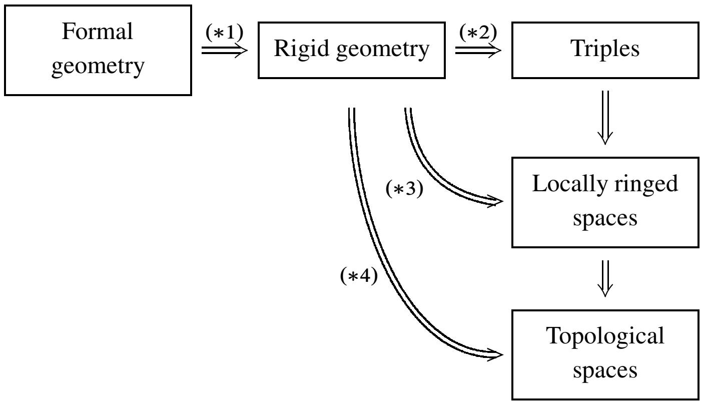

Figure 5. Birational approach to rigid geometry.

All this is the outline of what we will discuss in this volume. Here, before finishing this overview, let us add a few words on the outgrowth of our theory. Our approach to rigid geometry, in fact, gives rise to a new perspective of rigid geometry itself: *rigid geometry in general is an analysis along a closed subspace in a ringed topos*. This idea, which tells us what the concept of *rigid geometry* in mathematics should ultimately be, is linked with the idea of *tubular neighborhoods* in algebraic geometry, already discussed in [38]. From this viewpoint, Raynaud's choice, for example, of formal schemes as models of rigid spaces can be interpreted as capturing the 'tubular neighborhoods' along a closed subspace by means of the formal completion. Now that there are several other ways to capture such structures, e.g., Henselian schemes etc., there are several other choices for the model geometry of rigid geometry.[12] This yields several variants, e.g., rigid Henselian geometry, rigid Zariskian geometry, etc., all of which are encompassed within our birational approach.[13]

7. Relation with other theories. In the first three sections **II**, §A, **II**, §B, and **II**, §C of the appendices to Chapter **II**, we compare our theory with other theories related to rigid geometry. Here we give a digest of the contents of these sections for the reader's convenience.[14]

[12]There is, in addition to formal geometry and Henselian geometry, the third possibility for the model geometry, by *Zariskian schemes*. We provide a general account of the theory of Zariskian schemes and the associated rigid spaces, the so-called *rigid Zariskian spaces*, in the appendices **I**, §B and **II**, §D.

[13]The reader might note that this idea is also related to the cdh-topology in the theory of motivic cohomology.

[14]A. Abbes has recently published another foundational book [1] on rigid geometry, in which, similarly to ours, he developed and generalized Raynaud's approach to rigid geometry.

• **Relation with Tate's rigid analytic geometry.** Let V be an a-adically complete valuation ring of height one, and set $K = \mathrm{Frac}(V)$ (the fraction field), which is a complete non-Archimedean valued field with a non-trivial valuation $\|\cdot\|\colon K \to \mathbb{R}_{\geq 0}$. In **II**, §8.2. (c) we will define the notion of *classical points* (in the sense of Tate) for rigid spaces of a certain kind, including locally of finite type rigid spaces over $\mathcal{S} = (\mathrm{Spf}\, V)^{\mathrm{rig}}$. If $\mathcal{X}$ is a rigid space of the latter kind, it will turn out that the classical points of $\mathcal{X}$ are reduced zero-dimensional closed subvarieties in $\mathcal{X}$ (cf. **II**.8.2.6).

We define $\mathcal{X}_0$ to be the set of all classical points of $\mathcal{X}$. The assignment $\mathcal{X} \mapsto \mathcal{X}_0$ has several nice properties, some of which are incorporated into the notion of (*continuous*) *spectral functor* (cf. **II**, §8.1). Among them is the important property that classical points detect quasi-compact open subspaces: for quasi-compact open subspaces $\mathcal{U}, \mathcal{V} \subseteq \mathcal{X}$, $\mathcal{U}_0 = \mathcal{V}_0$ implies $\mathcal{U} = \mathcal{V}$. In view of all this, one can introduce on $\mathcal{X}_0$ a Grothendieck topology τ_0 and sheaf of rings $\mathcal{O}_{\mathcal{X}_0}$, which are naturally constructed from the topology and the structure sheaf of $\mathcal{X}$; for example, for a quasi-compact open subspace $\mathcal{U} \subseteq \mathcal{X}$, $\mathcal{U}_0$ is an admissible open subset of $\mathcal{X}_0$, and we have $\mathcal{O}_{\mathcal{X}_0}(\mathcal{U}_0) = \mathcal{O}_{\mathcal{X}}(\langle\mathcal{U}\rangle)$. It will turn out that the resulting triple $\mathcal{X}_0 = (\mathcal{X}_0, \tau_0, \mathcal{O}_{\mathcal{X}_0})$ is a Tate rigid analytic variety over K, and thus one has the canonical functor

$$\mathcal{X} \longmapsto \mathcal{X}_0$$

from the category of locally of finite type rigid spaces over $\mathcal{S}$ to the category of rigid analytic varieties over K.

Theorem (Theorem **II**.B.2.5, Corollary **II**.B.2.6). *The functor $\mathcal{X} \mapsto \mathcal{X}_0$ is a categorical equivalence from the category of quasi-separated locally of finite type rigid spaces over $\mathcal{S} = (\mathrm{Spf}\, V)^{\mathrm{rig}}$ to the category of quasi-separated Tate analytic varieties over K. Moreover, under this functor, affinoids* (*resp. coherent spaces*) *correspond to affinoid spaces* (*resp. coherent analytic spaces*).

Note that Raynaud's theorem (the existence of formal models) gives the canonical quasi-inverse functor to the above functor.[15]

• **Relation with Huber's adic geometry.** As we have already remarked above, the Zariski–Riemann triple $\mathbf{ZR}(\mathcal{X})$, at least in the situation as before, is an adic space. This is true in much more general situation, for example, in case $\mathcal{X}$ is *locally universally Noetherian* (**II**.2.2.23). In fact, by the enriched visualization, we have the functor

$$\mathbf{ZR}\colon \mathcal{X} \longmapsto \mathbf{ZR}(\mathcal{X})$$

[15] To show the theorem, we need the Gerritzen–Grauert theorem [45], which we assume whenever discussing Tate's rigid analytic geometry. Note that, when it comes to the rigid geometry over valuation rings, this volume is self-contained only with this exception. We will prove Gerritzen–Grauert theorem without a circular argument in the next volume.

from the category of locally universally Noetherian rigid spaces to the category of adic spaces (Theorem **II**.A.5.1), which gives rise to a categorical equivalence in the most important cases. In particular, we have the following theorem.

Theorem (Theorem **II**.A.5.2). *Let $\mathcal{S}$ be a locally universally Noetherian rigid space. Then* **ZR** *establishes a categorical equivalence from the category of locally of finite type rigid spaces over $\mathcal{S}$ to the category of adic spaces locally of finite type over* **ZR**$(\mathcal{S})$.

• **Relation with Berkovich analytic geometry.** Let V and K be as before. We will construct a natural functor

$$\mathcal{X} \longmapsto \mathcal{X}_{\mathrm{B}}$$

from the category of locally quasi-compact[16] (**II**.4.4.1) and locally of finite type rigid spaces over $\mathcal{S} = (\operatorname{Spf} V)^{\mathrm{rig}}$ to the category of strictly K-analytic spaces (in the sense of Berkovich).

Theorem (Theorem **II**.C.6.12). *The functor $\mathcal{X} \mapsto \mathcal{X}_{\mathrm{B}}$ establishes a categorical equivalence from the category of all locally quasi-compact locally of finite type rigid spaces over* $(\operatorname{Spf} V)^{\mathrm{rig}}$ *to the category of all strictly K-analytic spaces. Moreover, $\mathcal{X}_{\mathrm{B}}$ is Hausdorff* (*resp. paracompact Hausdorff, resp. compact Hausdorff*) *if and only if X is quasi-separated* (*resp. paracompact and quasi-separated, resp. coherent*).

The underlying topological space of $\mathcal{X}_{\mathrm{B}}$ is what we call the *separated quotient* (**II**, §4.3. (a)) of $\langle\mathcal{X}\rangle$, denoted by $[\mathcal{X}]$, which comes with the quotient map

$$\mathrm{sep}_{\mathcal{X}}\colon \langle\mathcal{X}\rangle \longrightarrow [\mathcal{X}]$$

(*separation map*). In particular, the topology of $\mathcal{X}_{\mathrm{B}}$ is the *quotient* topology of the topology of $\langle\mathcal{X}\rangle$.

Figure 6 illustrates the interrelations among the theories we have discussed so far. In the diagram,

- the functors $(*1)$ and $(*2)$ are fully faithful; the functor $(*3)$, defined on locally quasi-compact rigid analytic spaces, is fully faithful to the category of strictly K-analytic spaces;
- the functor $(*4)\colon \mathcal{X} \to \mathcal{X}_0$, defined on locally of finite type rigid spaces over $(\operatorname{Spf} V)^{\mathrm{rig}}$, is quasi-inverse to $(*1)$ restricted on quasi-separated spaces;

[16] Note that, if $\mathcal{X}$ is quasi-separated, then $\mathcal{X}$ is locally quasi-compact if and only if $\langle\mathcal{X}\rangle$ is taut in the sense of Huber, 5.1.2 in [61] (cf. **0**.2.5.6).

- the functor (∗5) is given by the enriched visualization, defined on locally universally Noetherian rigid spaces; it is fully faithful in practical situations, including those of locally of finite type rigid spaces over a fixed locally universally Noetherian rigid space, and of rigid spaces of type (N) (**II**.A.5.3);
- the functor (∗6): $\mathscr{X} \mapsto \mathscr{X}_{\mathrm{B}}$, defined on locally quasi-compact locally of finite type rigid spaces over $(\operatorname{Spf} V)^{\mathrm{rig}}$, establishes a categorical equivalence with the category of strictly K-analytic spaces.

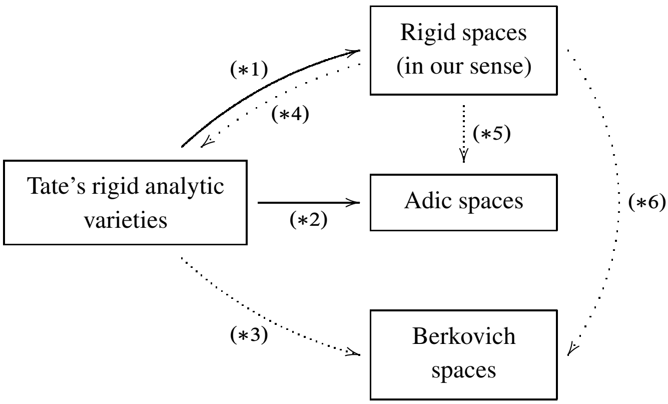

Figure 6. Relation with other theories.

Finally, we would like to mention that it has recently become known to experts in the field that it is possible that some of the non-Archimedean spaces that arise naturally in contemporary arithmetic geometry cannot be handled in Berkovich's analytic geometry (see e.g. [57], 4.4). This state of affair makes it important to investigate in detail the relationship between Berkovich's analytic geometry and rigid geometry (or adic geometry). In **II**, §C.5, we will study a spectral theory of filtered rings and introduce a new category of spaces, the so-called metrized analytic spaces. This new notion of spaces generalizes Berkovich's K-analytic spaces, and gives a clear picture of the comparison; see **II**, §C.6. (d). Also, the newly introduced spaces turn out to be equivalent to Kedlaya's reified adic spaces [67], to which our filtered ring approach in this book offers a new perspective.

8. Applications. We expect that our rigid geometry will have rich applications, not only in arithmetic geometry, but also in various other fields. A few of them have already been sketched in [42], which include

- arithmetic moduli spaces (e.g. Shimura varieties) and their compactifications;
- trace formula in characteristic $p > 0$ (Deligne's conjecture).

In addition to these, since our theory has set out from Zariski's birational geometry, applications to problems in birational geometry, modern or classical, are also expected. For example, this volume already contains Nagata's compactification theorem for schemes and a proof of it (**II**, §F), as an application of the general idea of our rigid geometry to algebraic geometry.

Some other prospective applications may be to p-adic Hodge theory (cf. [91] and [92]) and to rigid cohomology theory for algebraic varieties in positive characteristic. Here the visualization in our sense of rigid spaces will give concrete pictures for tubes and the dagger construction. As one application in this direction one can mention

- p-adic Hodge theory vs. rigid cohomology.

Finally, let us mention that the applications to

- moduli of Galois representations,
- mirror symmetry,

the second of which has been first envisaged by M. Kontsevich, should be among the future challenges.

9. Contents of this book. We followed two basic rules in designing the contents of this book, both of which may justify its length. First, in addition to being a front-line exposition presenting new theories and results, we hope that this book will serve as an encyclopedic source. It contains, consequently, as many notions and concepts, hopefully with only few omission, that should come about as basic and important ones for present and future use, as possible.

Second, we have aimed at making this book as self-contained as possible. All results that sit properly inside the main body of our arguments are always followed by proofs, except for some minor or not-too-difficult lemmas, some of which are placed at the end of each section as exercises; even in this case, if the result is used in the main text, we give a detailed hint in the end of the book, which, in many cases, almost proves the assertion. Note that, because of several laborious requirements on the groundwork, such as removing the Noetherian hypothesis, are also self-contained many of the preliminary parts.

This volume consists of the following three chapters:

- Chapter **0**. Preliminaries
- Chapter **I**. Formal geometry
- Chapter **II**. Rigid spaces

Let us briefly sketch the contents of each chapter. More detailed summaries will be given at the beginning of each chapter.

Chapter **0** collects preliminaries, which, however, contain also new results. Sections **0**, §1 to **0**, §7 give necessary preliminaries on set theory, category theory, general topology, homological algebra, etc. In the general topology section, we put emphasis on Stone duality between topological spaces and lattices. In **0**, §8 and **0**, §9, we will conduct thorough study of topological and algebraic aspects of topological rings and modules. This part of the preliminaries will be the bases of the next chapter, the general theory of formal geometry.

Chapter **I** is devoted to formal geometry. The essential task here is to treat non-Noetherian formal schemes of a certain kind, e.g., finite type formal schemes over an a-adically complete valuation ring of arbitrary height, for reasons of functoriality (as stated in **4.** above). Since this kind of generalities seem to be missing in the past literature, we provide a self-contained and systematic theory of formal geometry, generalizing many of the theorems in [54], **III**. To this end, we introduce several new notions of finiteness condition *outside the ideal of definition* and show that they allow one to build a versatile theory of formal schemes.

Chapter **II** is the main part of this volume, in which we develop rigid geometry, based on the foundational work done in the previous chapter. The geometrical theory of rigid spaces that we treat in this chapter includes

- cohomology theory of coherent sheaves (**II**, §5, §6); finiteness (**II**.7.5.19);
- local and global study of morphisms (**II**, §7);
- classification of points (**II**, §8, §11.1);
- GAGA (**II**, §9);
- relations with other theories (**II**, §A, §B, §C).

There are of course many other important topics that are not dealt with in this volume. Some of them, including several important applications, will be contained in the future volumes.

10. Use of algebraic spaces. In **I**, §6 we develop a full-fledged theory of formal algebraic spaces. It is, in fact, one of the characteristic features of our approach to rigid geometry that we allow formal algebraic spaces, not only formal schemes, to be formal models of rigid spaces. The motivation mainly comes from the applications to algebraic geometry.

In algebraic geometry, while it is often difficult to show that spaces, such as moduli spaces, are represented by schemes, the representability by algebraic spaces is relatively easy to establish, thanks to M. Artin's formal algebraization theorem [6]. Therefore, taking algebraic spaces into the scope increases the flexibility of the theory. In order to incorporate algebraic spaces into our rigid geometry, one first needs to discuss formal algebraic spaces, some of which appear as the formal completion of algebraic spaces, and then proceed to the rigid spaces associated to them. Now the important fact is that, although formal algebraic spaces seem to constitute, via Raynaud's recipe, a new category of rigid spaces that enlarges the already existing category of rigid spaces derived from formal schemes, they actually do not; viz., we do not have to enlarge the category of rigid spaces by this generalization. This is explained by the following theorem, which we shall prove in the future volume.

Theorem (equivalence theorem). *Let X be a coherent adic formal algebraic space of finite ideal type. Then there exists an admissible blow-up $X' \to X$ from a formal scheme X'. Therefore, the canonical functor*

$$\left\{\begin{array}{c}\text{coherent adic}\\ \text{formal schemes of}\\ \text{finite ideal type}\end{array}\right\}\Big/\left\{\begin{array}{c}\text{admissible}\\ \text{blow-ups}\end{array}\right\} \longrightarrow \left\{\begin{array}{c}\text{coherent adic}\\ \text{formal algebraic}\\ \text{spaces of}\\ \text{finite ideal type}\end{array}\right\}\Big/\left\{\begin{array}{c}\text{admissible}\\ \text{blow-ups}\end{array}\right\}$$

is a categorical equivalence.

The theorem shows that, up to admissible blow-ups, formal algebraic spaces simply fall into the class of formal schemes, and thus define the associated rigid space X^{rig} just 'as usual.' As for GAGA, we can generalize the definition of GAGA functor for algebraic spaces (using a compactification theorem of Nagata type for algebraic spaces).[17]

11. Properness in rigid geometry. In rigid geometry, we have the following three natural definitions of properness. A morphism $\varphi\colon \mathcal{X} \to \mathcal{Y}$ of coherent rigid spaces is *proper* if either one of the following conditions is satisfied.

(1) φ is universally closed (**II**.7.5.4), separated, and of finite type.

(2) *Raynaud properness.* There exists a proper formal model $f\colon X \to Y$ of φ.

(3) *Kiehl properness.* φ is separated of finite type, and there exist an affinoid covering $\{\mathcal{U}_i\}_{i\in I}$ and, for each $i \in I$, a pair of finite affinoid coverings $\{\mathcal{V}_{ij}\}_{j\in J_i}$ and $\{\mathcal{V}'_{ij}\}_{j\in J_i}$ of $\varphi^{-1}(\mathcal{U}_i)$ indexed by a common set J_i such that, for any $j \in J_i$, $\mathcal{V}_{ij} \subseteq \mathcal{V}'_{ij}$ and $\mathcal{V}_{ij}$ is relatively compact in $\mathcal{V}'_{ij}$ over $\mathcal{U}_i$ (in the sense of Kiehl).

[17] This 'analytification of algebraic spaces' was already considered in depth and developed by B. Conrad and M. Temkin [31] over complete non-Archimedean valued fields.

Historically, properness in Tate's rigid geometry has been first defined by R. Kiehl by condition (3) in his work [69] on finiteness theorem. This condition, existence of affinoid enlargements, stems from the general idea by Cartan and Serre and by H. Grauert for proving finiteness of cohomologies of coherent sheaves. While the equivalence of (1) and (2) is an easy exercise, the equivalence of (2) and (3), especially the implication (2) $\Longrightarrow$ (3), is a very deep theorem. Lütkebohmert's 1990 paper [78] proves this for rigid spaces of finite type over $(\mathrm{Spf}\, V)^{\mathrm{rig}}$, where V is a complete discrete valuation ring. In this book, we temporarily define properness by condition (1) (and hence equivalently by (2)), and postpone the proof of the equivalence of these three conditions, especially the implication (2) $\Longrightarrow$ (3), in the so-called *adhesive* case (**II**.2.2.23), to the next volume, in which we will show the *Enlargement Theorem* by expanding Lütkebohmert's technique.

12. Contents of the future volumes. Our project will continue in future volumes. The next volume will contain the following chapters.

- **Chapter III. Formal flattening theorem**

  This chapter will also contain several applications of the formal flattening theorem, such as Gerritzen–Grauert theorem.

- **Chapter IV. Enlargement theorem**

  This chapter will contain the proof of the equivalence of the three 'definitions' of properness.

- **Chapter V. Equivalence theorem and analytification of algebraic spaces**

  This chapter will give the proof of the equivalence theorem stated above and the definition of the GAGA functor for algebraic spaces.

13. General conventions. Chapter numbers are bold-face Roman, while for sections and subsections we use Arabic numbers; subsubsections are numbered by letters in parentheses; for example, '**I**, §3.2. (b)' refers to the second subsubsection of the second subsection in §3 of Chapter **I**. Cross-references will be given by sequences of numerals, like **I**.3.2.1, which specify the places of the statements in the text. The chapter numbers are omitted when referring to places in the same chapter.

Almost all sections are equipped with some exercises at the end, which are selected in order to help the reader understand the content. We insert hints for some of the exercises at the end of this volume.

Let us list some mathematical conventions.

- We fix once for all a Grothendieck universe U ([8], Exposé I, 0); cf. **0**, §1.1. (a).
- By a Grothendieck topology (or simply by a topology) on a category $\mathscr{C}$ we always mean a functor $J: x \mapsto J(x)$, assigning to each $x \in \mathrm{obj}(\mathscr{C})$ a collection of sieves, as in [80], III, §2, Definition 1. In many places, however, Grothendieck topologies are introduced by means of a base (covering families) as in [80], III, §2, Definition 2, (*prétopologie* in the terminology in [8], Exposé II, (1.3)); in this situation, we consider, without explicit mentioning, the Grothendieck topology generated by the base.
- A site will always mean a U-*site* (cf. [8], Exposé II, (3.0.2)), that is, a pair $(\mathscr{C}, J)$ consisting of a U-category $\mathscr{C}$ ([8], Exposé I, Definition 1.1) and a Grothendieck topology on $\mathscr{C}$.
- All compact topological spaces are assumed to be Hausdorff; that is, we adopt the Bourbaki convention

  $$\text{quasi-compact} + \text{Hausdorff} = \text{compact}.$$

  However, we sometimes use the term 'compact Hausdorff' just for emphasis. Other conventions, in which we do *not* follow Bourbaki, are the following ones.

  - Locally compact spaces are only assumed to be *locally* Hausdorff;[18] A topological space X is said to be *locally compact* if every point of X has a compact neighborhood contained in a Hausdorff neighborhood.
  - Paracompact spaces are *not* assumed to be Hausdorff; see **0**, §2.5. (c).
- Whenever we say A is a ring, we always mean, unless otherwise stipulated that A is a commutative ring having the multiplicative unit $1 = 1_A$. We also assume that any ring homomorphism $f: A \to B$ is unital, that is, maps 1_A to 1_B. Moreover,
  - for a ring A we denote by $\mathrm{Frac}(A)$ the total ring of fractions of A;
  - for a ring A the Krull dimension of A is denoted by $\dim(A)$;
  - when A is a local ring, its unique maximal ideal is denoted by $\mathfrak{m}_A$.
- Let A be a ring and $I \subseteq A$ an ideal. When we say A is I-adically complete or complete with respect to the I-adic topology, we always mean, unless otherwise stipulated that A is *Hausdorff complete* with respect to the I-adic topology.

[18] Note that, in [24], Chapter I, §9.7, Definition 4, locally compact spaces are assumed to be Hausdorff.

- By an exact functor between derived categories (of any sort) we always mean an exact functor of triangulated categories that preserves the canonical t-structures (hence also the canonical cohomology functors), which are clearly specified by the context.
- We will often use, by abuse of notation, the equality symbol '$=$' for 'isomorphic by a canonical morphism.'

Acknowledgements. There are so many people and institutions to whom we are deeply indebted for help with this work. We first of all would like to thank Ofer Gabber for many valuable and inspiring discussions and results, partly obtained by collaboration, which improved very much our original results and initial blue-prints of this project. We thank Ahmed Abbes, Yves André, Matt Baker, Ken-ichi Bannai, Francesco Baldassarri, Vladimir Berkovich, Siegfried Bosch, Bruno Chiarellotto, Gunther Cornelissen, Quentin Guignard, Michel Gros, Bernard Le Stum, Yoichi Mieda, Johannes Nicaise, Frans Oort, Michel Raynaud, Takeshi Saito, Peter Ullrich, and Gerard van der Geer for valuable discussions and friendly encouragement. We are grateful to Institut des Hautes Études Scientifiques, Research Institute for Mathematical Sciences, Centre de Recerca Matemàtica, École Normale Supérieure, Université de Rennes 1, Universiteit Utrecht, and Nagoya University for their hospitality. Finally, let us mention that this work was supported by JSPS KAKENHI Grant Numbers 13440004, 17340002, 21340004, 17740014, 20540015, 23540015, 26400050, and 15H03607.