Contents

Preface . . . . . *v*

Frequently used notation . . . . . *xv*

1. Brief introduction to linear forms in logarithms 1
   - 1.1. Linear forms in complex logarithms 1
   - 1.2. Linear forms in p-adic logarithms 6
   - 1.3. Linear forms in elliptic logarithms 6

2. Lower bounds for linear forms in complex and p-adic logarithms 9
   - 2.1. Lower bounds for linear forms in complex logarithms 9
   - 2.2. Multiplicative dependence relations between algebraic numbers 17
   - 2.3. Lower bounds for linear forms in p-adic logarithms 17
   - 2.4. Notes 22

3. First applications 23
   - 3.1. On the distance between powers of 2 and powers of 3 23
   - 3.2. Effective irrationality measures for quotients of logarithms of integers . . 25
   - 3.3. On the distance between two integral S-units 25
   - 3.4. Effective irrationality measures for n-th roots of algebraic numbers 26
   - 3.5. On the greatest prime factor of values of integer polynomials 27
   - 3.6. On the greatest prime factor of terms of linear recurrence sequences . . . 30
   - 3.7. Perfect powers in linear recurrence sequences 32
   - 3.8. Simultaneous Pellian equations and Diophantine quadruples 35
   - 3.9. On the representation of integers in distinct bases 39
   - 3.10. Further applications (without proofs) 43
   - 3.11. Exercises 44
   - 3.12. Notes 44

4. Classical families of Diophantine equations 47
   - 4.1. Proof of Baker's Theorem 1.10 47
   - 4.2. The unit equation 50
   - 4.3. The Thue equation 52
   - 4.4. Effective improvement of Liouville's inequality 55
   - 4.5. The superelliptic and hyperelliptic equations 55
   - 4.6. The Diophantine equation $x^2 + C = y^n$ 57
   - 4.7. Perfect powers at integer values of polynomials 60

4.8. The Catalan equation and the Pillai conjecture . . . 62
4.9. Exercises . . . 65
4.10. Notes . . . 65

5. Further applications . . . 67
5.1. Effective irrationality measures for quotients of logarithms of rational numbers . . . 67
5.2. Effective irrationality measures for n-th roots of rational numbers . . . 68
5.3. The Thue equation $ax^n - by^n = c$ . . . 69
5.4. A generalization of Diophantine quadruples . . . 71
5.5. Exercises . . . 72
5.6. Notes . . . 73

6. Applications of linear forms in p-adic logarithms . . . 75
6.1. On the p-adic distance between two integral S-units . . . 75
6.2. Waring's problem . . . 76
6.3. On the b-ary expansion of an algebraic number . . . 77
6.4. Repunits and perfect powers . . . 80
6.5. Perfect powers with few binary digits . . . 81
6.6. The S-unit equation . . . 83
6.7. The Thue–Mahler equation and other classical equations . . . 85
6.8. Perfect powers in binary recurrence sequences . . . 86
6.9. Perfect powers as sum of two integral S-units . . . 88
6.10. On the digital representation of integral S-units . . . 89
6.11. Exercises . . . 90
6.12. Notes . . . 92

7. Primitive divisors and the greatest prime factor of $2^n - 1$ . . . 95
7.1. Primitive divisors . . . 95
7.2. Primitive divisors of Lucas–Lehmer sequences . . . 98
7.3. The Diophantine equation $x^2 + C = y^n$, continued . . . 101
7.4. On the number of solutions to the Diophantine equation $x^2 + D = p^n$ . . . 102
7.5. On the greatest prime factor of $2^n - 1$ . . . 103
7.6. Exercises . . . 106
7.7. Notes . . . 106

8. The abc-conjecture . . . 109
8.1. Effective results towards the abc-conjecture . . . 109
8.2. Proofs of Theorems 8.2 and 8.3 . . . 110
8.3. Exercise . . . 115
8.4. Notes . . . 115

9. Simultaneous linear forms in logarithms and applications . . . 117
9.1. A theorem of Loxton . . . 117
9.2. Perfect powers in short intervals . . . 118
9.3. Simultaneous Pellian equations with at most one solution . . . 121
9.4. Exercise . . . 123
9.5. Notes . . . 123

10. Multiplicative dependence relations between algebraic numbers 125
10.1. Lower bound for the height of an algebraic number 125
10.2. Existence of small multiplicative dependence relations 127

11. Lower bounds for linear forms in two complex logarithms: proofs 131
11.1. Three estimates for linear forms in two complex logarithms 131
11.2. An auxiliary inequality involving several parameters 132
11.3. Proof of Theorem 11.4 133
11.4. Deduction of Theorem 11.1 from Theorem 11.4 138
11.5. Deduction of Theorem 11.2 from Theorem 11.4 140

12. Lower bounds for linear forms in two p-adic logarithms: proofs 143
12.1. Three estimates for linear forms in two p-adic logarithms 143
12.2. Two auxiliary inequalities involving several parameters 145
12.3. Proofs of Theorems 12.4 and 12.5 147
12.4. Deduction of Theorem 12.1 from Theorem 12.4 153
12.5. Deduction of Theorem 12.2 from Theorem 12.5 155
12.6. Deduction of Theorem 12.3 from Theorem 12.1 156

13. Open problems 157
13.1. Classical conjectures in transcendence theory 157
13.2. Diophantine equations 158
13.3. Miscellaneous 161

Appendices 165

A. Approximation by rational numbers 167

B. Heights 171
B.1. Definitions 171
B.2. The Liouville inequality 176
B.3. Linear forms in one p-adic logarithm 177

C. Auxiliary results on algebraic number fields 179
C.1. Real quadratic fields and Pellian equations 179
C.2. Algebraic number fields 180

D. Classical results on prime numbers 183

E. A zero lemma 187

F. Tools from complex and p-adic analysis 191
F.1. The Schwarz lemma in complex analysis 191
F.2. Auxiliary results on p-adic fields 191
F.3. The Schwarz lemma in p-adic analysis 193

Bibliography 195

Index 223