Preface

The interest for Kac–Moody algebras and groups has grown exponentially in the past decades, both in the mathematical and physics communities. In physics, this interest has essentially been focused on *affine* Kac–Moody algebras and groups (see e.g. [Kac90]), until the recent development of *M-theory*, which also brought into the game certain Kac–Moody algebras and groups of *indefinite* type (see e.g., [DHN02], [DN05], [FGKP18]). Within mathematics, Kac–Moody groups have been studied from a wide variety of perspectives, reflecting the variety of flavours in which they appear: as for the group functor SL_n, which associates to each field $\mathbb{K}$ the group $\mathrm{SL}_n(\mathbb{K}) = \{A \in \mathrm{Mat}_n(\mathbb{K}) \mid \det A = 1\}$, Kac–Moody groups can be constructed over any field $\mathbb{K}$. In addition, Kac–Moody groups come in two versions (*minimal* and *maximal*). To a given Kac–Moody algebra is thus in fact attached a family of groups, whose nature can greatly vary. Just to give a glimpse of this variety, here is a (neither exhaustive nor even representative, and possibly random) list of recent research directions.

Note first that any Kac–Moody group naturally acts (in a nice way) on some geometric object, called a *building*. Buildings have an extensive theory of their own (see [AB08]) and admit several metric realisations, amongst which CAT(0)-*realisations*. In turn, CAT(0)-spaces have been extensively studied (see [BH99]). This already provides powerful machineries to study Kac–Moody groups, and connects Kac–Moody theory to many topics of geometric group theory.

Over $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$, minimal Kac–Moody groups G are connected Hausdorff topological groups. In [FHHK17], symmetric spaces (in the axiomatic sense of Loos) associated to G are defined and studied. In [Kit14], cohomological properties of the *unitary form* K of G (i.e. the analogue of a maximal compact subgroup in $\mathrm{SL}_n(\mathbb{C})$) are investigated. Maximal Kac–Moody groups over $\mathbb{K} = \mathbb{C}$, on the other hand, have a rich algebraic-geometric structure (see e.g., [Mat88b], [Kum02], [Pez17]).

Over a non-Archimedean local field $\mathbb{K}$, the authors of [GR14] associate *spherical Hecke algebras* to Kac–Moody groups of arbitrary type, using a variant of buildings, called *hovels*.

When $\mathbb{K}$ is a finite field, minimal Kac–Moody groups provide a class of discrete groups that combine various properties in a very singular way. For instance, they share many properties with arithmetic groups (see [Rém09]), but are typically simple; they in fact provide the first infinite finitely presented examples of discrete groups that are both simple and Kazhdan (see [CR09]). They also helped construct Golod–Shafarevich groups that disproved a conjecture by E. Zelmanov (see [Ers08]).

Maximal Kac–Moody groups over finite fields, on the other hand, provide an important family of simple (non-discrete) totally disconnected locally compact groups (see [Rém12], [CRW17], and also §9.4).

Despite the manifold attractions of general Kac–Moody groups, the vast majority of the works in Kac–Moody theory still focus on affine Kac–Moody groups. We strongly believe that this is in part due to the absence of an introductory textbook on the subject (apart from Kumar's book [Kum02] which, however, only covers the case $\mathbb{K} = \mathbb{C}$), which can make learning about general Kac–Moody groups a long and difficult journey. The present book was born out of the desire to fill this gap in the literature, and to provide an accessible, intuitive, reader-friendly, self-contained and yet concise introduction to Kac–Moody groups. It also aims at "cleaning" the foundations and providing a unified treatment of the theory. The targeted audience includes anyone interested in learning about Kac–Moody algebras and/or groups (with a minimal background in linear algebra and basic topology — this book actually grew out of lecture notes for a Master course on Kac–Moody algebras and groups), as well as more seasoned researchers and experts in Kac–Moody theory, who may find in this book some clarifications for the many rough spots of the current literature on Kac–Moody groups. A description of the structure of the book, as well as a guide to the reader, are provided at the end of the introduction.

To conclude, some acknowledgements are in order. I am very much indebted to Guy Rousseau, first for his paper [Rou16] which made it possible for me to write Chapter 8 of this book, and second for his thorough comments on an earlier version of that chapter. I am also indebted to Pierre-Emmanuel Caprace, for introducing me to the world of Kac–Moody groups in the first place, and for his precious comments on an earlier version of Chapter 7. Finally, I extend my warmest thanks to Ralf Köhl and anonymous reviewers for their precious comments on an earlier version of the book. Needless to say, all remaining mistakes are entirely mine.

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