Introduction

The story begins with fundamental works of S. Lie, W. Killing, and E. Cartan back at the end of the 19th century, which led to the creation of *Lie theory*. Lie's original motivation was to develop a Galois theory for differential equations; by now, Lie theory has gone far beyond this objective, and has become a central chapter of contemporary mathematics. At its heart lies the study of certain groups of symmetry of algebraic or geometric objects (the *Lie groups*), of their corresponding sets of infinitesimal transformations (their *Lie algebras*), and of the fruitful interplay between Lie groups and Lie algebras (the *Lie correspondence*).

To any Lie group G, one can namely associate the vector space $\mathfrak{g}$ of its tangent vectors at the identity, and equip $\mathfrak{g}$ with a Lie bracket $\mathfrak{g} \times \mathfrak{g} \to \mathfrak{g} : (x, y) \mapsto [x, y]$, turning $\mathfrak{g}$ into a Lie algebra. For our purposes, it will be sufficient to think of $G = \mathrm{SL}_n(\mathbb{C})$, in which case $\mathfrak{g} = \mathfrak{sl}_n(\mathbb{C})$ is the Lie algebra of traceless complex $n \times n$ matrices, with the Lie bracket given by $[A, B] := AB - BA$ for all $A, B \in \mathfrak{g}$.

Conversely, the Lie group G can be reconstructed from its Lie algebra $\mathfrak{g}$ by exponentiation: the *exponential map*

$$\exp \colon \mathfrak{sl}_n(\mathbb{C}) \to \mathrm{SL}_n(\mathbb{C}) : A \mapsto e^A := \sum_{n \geq 0} \frac{1}{n!} A^n$$

yields an identification of the underlying sets, and the group multiplication on G can be expressed in terms of the Lie bracket. The significance of this Lie correspondence is that one can basically understand the group G by studying the linear (hence simpler) object $\mathfrak{g}$.

Many interesting examples of "transformation groups" yield *finite-dimensional* Lie groups, that is, Lie groups whose Lie algebra has finite dimension as a vector space. The class of finite-dimensional complex Lie algebras has been extensively studied since the creation of Lie theory, and the classification by Killing and Cartan of its simple pieces (the *simple* Lie algebras) is arguably one of the greatest mathematical achievements from around the turn of the twentieth century. This classification yields a small list of simple Lie algebras (of which $\mathfrak{sl}_n(\mathbb{C})$ is an example), indexed by some matrices of integers A (the *Cartan matrices*).

The path to infinite-dimensional Lie algebras and associated groups, on the other hand, is far less unique, and there is at present no general theory for these objects. Their study also began much later, around the late 1960's, and one can distinguish two general directions: one more analytic, investigating Lie groups modelled on infinite-dimensional spaces such as Banach or Fréchet spaces, as in [Nee06], and the other more algebraic, leading to Kac–Moody theory.

By a theorem of J.-P. Serre ([Ser66]), any finite-dimensional (semi-)simple Lie algebra admits a presentation (i.e. a definition by generators and relations)

whose parameters are the entries of the corresponding Cartan matrix A. Now, this presentation still makes sense if one allows more general integral matrices A, called *generalised Cartan matrices*. The corresponding Lie algebras (the *Kac–Moody algebras*) were introduced independently in 1967 by V. Kac (whose original motivation was to classify certain symmetric spaces, see [Kac67], [Kac68]) and R. Moody ([Moo67], [Moo68]). They share many properties with their (finite-dimensional) older sisters, but also show some striking differences. These differences account for a very rich theory of *Kac–Moody groups* (i.e. of groups associated to a Kac–Moody algebra), with the apparition of new phenomena that are absent from the classical theory.

We give below a brief outline of the story that this book is trying to tell, starting from finite-dimensional simple Lie algebras, and moving towards the construction of objects deserving the name of "Kac–Moody groups".

1 Finite-dimensional simple Lie algebras

Let $\mathfrak{g}$ be a finite-dimensional simple Lie algebra, such as $\mathfrak{g} = \mathfrak{sl}_n(\mathbb{C})$ (precise definitions for the terminology used in this section will be given in Chapter 2). Thus $\mathfrak{g}$ is a complex vector space with a Lie bracket $[\cdot, \cdot]$, which is encoded in the *adjoint representation*

$$\mathrm{ad}: \mathfrak{g} \to \mathrm{End}(\mathfrak{g}), \quad \mathrm{ad}(x)y := [x, y] \quad \text{for all } x, y \in \mathfrak{g}$$

of $\mathfrak{g}$ on itself.

The first step in trying to understand the structure of $\mathfrak{g}$ is to prove the existence of a *Cartan subalgebra* $\mathfrak{h}$ of $\mathfrak{g}$, namely, of a nontrivial subalgebra $\mathfrak{h}$ all whose elements h are ad-diagonalisable (i.e. $\mathrm{ad}(h) \in \mathrm{End}(\mathfrak{g})$ is diagonal in some suitable basis of $\mathfrak{g}$) and that is maximal for this property. Then the elements of $\mathfrak{h}$ are simultaneously ad-diagonalisable: in other words, $\mathfrak{g}$ admits a *root space decomposition*

$$\mathfrak{g} = \bigoplus_{\alpha \in \mathfrak{h}^*} \mathfrak{g}_\alpha, \tag{1}$$

where

$$\mathfrak{g}_\alpha := \{x \in \mathfrak{g} \mid [h, x] = \alpha(h)x \ \forall h \in \mathfrak{h}\}$$

is the α-eigenspace of $\mathrm{ad}(\mathfrak{h})$. The nonzero elements $\alpha \in \mathfrak{h}^*$ such that $\mathfrak{g}_\alpha \neq \{0\}$ are called *roots*, and their set is denoted Δ. One shows that $\mathfrak{g}_0 = \mathfrak{h}$, so that (1) may be rewritten as

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta} \mathfrak{g}_\alpha. \tag{2}$$

Example 1. Let $\mathfrak{g} = \mathfrak{sl}_n(\mathbb{C})$, and write E_{ij} for the $n \times n$ matrix with an entry "1" in position (i, j) and "0" elsewhere. The subalgebra

$$\mathfrak{h} := \operatorname{span}_{\mathbb{C}}\langle \alpha_i^{\vee} := E_{ii} - E_{i+1,i+1} \mid 1 \le i \le n - 1\rangle$$

of all diagonal matrices in $\mathfrak{sl}_n(\mathbb{C})$ is a Cartan subalgebra: the ad-diagonalisability of $\mathfrak{h}$ follows from the computation

$$[\alpha_i^{\vee}, E_{jk}] = (\delta_{ij} - \delta_{ik} - \delta_{i+1,j} + \delta_{i+1,k})E_{jk} = (\varepsilon_j - \varepsilon_k)(\alpha_i^{\vee})E_{jk} \quad \text{for all } i, j, k,$$

where $\varepsilon_j(E_{ii}) := \delta_{ij}$. The corresponding set of roots and root spaces are then given by

$$\Delta = \{\alpha_{jk} := \varepsilon_j - \varepsilon_k \mid 1 \le j \ne k \le n\} \quad \text{and} \quad \mathfrak{g}_{\alpha_{jk}} = \mathbb{C}E_{jk},$$

yielding the root space decomposition $\mathfrak{sl}_n(\mathbb{C}) = \mathfrak{h} \oplus \bigoplus_{j \ne k} \mathbb{C}E_{jk}$.   □

The second step is to establish some properties of the $\mathfrak{g}_\alpha$'s. Here are some important ones:

(1) $\dim \mathfrak{g}_\alpha = 1$ for all $\alpha \in \Delta$.

(2) For any nonzero $x_\alpha \in \mathfrak{g}_\alpha$ ($\alpha \in \Delta$), there is some $x_{-\alpha} \in \mathfrak{g}_{-\alpha}$ such that the assignment

$$x_\alpha \mapsto \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad x_{-\alpha} \mapsto \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}, \quad \alpha^{\vee} := [x_{-\alpha}, x_\alpha] \in \mathfrak{h} \mapsto \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

defines an isomorphism $\mathfrak{g}_{(\alpha)} := \mathbb{C}x_{-\alpha} \oplus \mathbb{C}\alpha^{\vee} \oplus \mathbb{C}x_\alpha \to \mathfrak{sl}_2(\mathbb{C})$ of Lie algebras.

(3) $\alpha(\beta^{\vee}) \in \mathbb{Z}$ for all $\alpha, \beta \in \Delta$.

The third step is to study the root system Δ and to show that, together with the integers $\alpha(\beta^{\vee})$ ($\alpha, \beta \in \Delta$), it completely determines $\mathfrak{g}$. Actually, Δ admits a *root basis* $\Pi = \{\alpha_1, \ldots, \alpha_\ell\}$ (i.e. every $\alpha \in \Delta$ can be uniquely expressed as a linear combination $\alpha = \pm \sum_{i=1}^{\ell} n_i \alpha_i$ for some $n_i \in \mathbb{N}$), and $\mathfrak{g}$ is already uniquely determined by the *Cartan matrix*

$$A = (a_{ij})_{1 \le i,j \le \ell} := (\alpha_j(\alpha_i^{\vee}))_{1 \le i,j \le \ell}.$$

More precisely, choosing elements $e_i = x_{\alpha_i} \in \mathfrak{g}_{\alpha_i}$ and $f_i = x_{-\alpha_i} \in \mathfrak{g}_{-\alpha_i}$ as above, $\mathfrak{g}$ is generated by the ℓ copies $\mathfrak{g}_{(\alpha_i)} := \mathbb{C}f_i \oplus \mathbb{C}\alpha_i^{\vee} \oplus \mathbb{C}e_i$ of $\mathfrak{sl}_2(\mathbb{C})$ ($1 \le i \le \ell$), and can even be reconstructed as the complex Lie algebra $\mathfrak{g}_A$ on the 3ℓ generators $e_i, f_i, \alpha_i^{\vee}$ and with the following defining relations ($1 \le i, j \le \ell$):

$$[\alpha_i^{\vee}, \alpha_j^{\vee}] = 0, \ [\alpha_i^{\vee}, e_j] = a_{ij}e_j, \ [\alpha_i^{\vee}, f_j] = -a_{ij}f_j, \ [f_i, e_j] = \delta_{ij}\alpha_i^{\vee}, \quad (3)$$

$$(\operatorname{ad} e_i)^{1-a_{ij}} e_j = 0, \quad (\operatorname{ad} f_i)^{1-a_{ij}} f_j = 0 \qquad \text{for } i \ne j. \tag{4}$$

Note that the relations (4), called the *Serre relations*, make sense, as the $a_{ij} \in \mathbb{Z}$ in fact satisfy $a_{ij} \le 0$ whenever $i \ne j$.

Example 2. We keep the notations of Example 1. For each $j, k \in \{1, \ldots, n\}$ with $j \neq k$, we get an embedded copy of $\mathfrak{sl}_2(\mathbb{C})$ in $\mathfrak{sl}_n(\mathbb{C})$ by considering submatrices indexed by $\{j, k\}$. One can then take

$$x_{\alpha_{jk}} := E_{jk} \in \mathfrak{g}_{\alpha_{jk}}, \quad x_{-\alpha_{jk}} := -E_{kj} \in \mathfrak{g}_{-\alpha_{jk}} \quad \text{and} \quad \alpha_{jk}^\vee := E_{jj} - E_{kk}.$$

We set $\alpha_i := \alpha_{i,i+1}$ for each $i \in \{1, \ldots, n-1\}$, so that $\alpha_i^\vee = E_{ii} - E_{i+1,i+1}$ is consistent with our previous notations. Then $\Pi = \{\alpha_i \mid 1 \leq i \leq n-1\}$ is indeed a root basis of Δ, and $\mathfrak{sl}_n(\mathbb{C})$ is generated, as a Lie algebra, by the elements $e_i := E_{i,i+1}$ and $f_i := -E_{i+1,i}$ $(1 \leq i \leq n-1)$. The Cartan matrix $A = (\alpha_j(\alpha_i^\vee))_{1 \leq i,j \leq n-1}$ has 2's on the main diagonal, -1's on the diagonals $(i, i+1)$ and $(i+1, i)$, and 0's elsewhere.   □

2 Kac–Moody algebras

To define infinite-dimensional generalisations of the simple Lie algebras (and, later on, of the simple Lie groups), we follow the opposite path to the one leading to the classification of simple Lie algebras (and groups): we start from "generalised" Cartan matrices A, then define a Lie algebra associated to A, and then, eventually, a group associated to this Lie algebra.

More precisely, the presentation of the Lie algebra $\mathfrak{g}_A$ introduced in the previous section still makes sense if $A = (a_{ij})_{1 \leq i,j \leq \ell}$ is a *generalised Cartan matrix* (GCM), in the sense that, for each $i, j \in \{1, \ldots, \ell\}$,

(C1) $a_{ii} = 2$ (to ensure that $e_i, f_i, \alpha_i^\vee$ span a copy of $\mathfrak{sl}_2(\mathbb{C})$),

(C2) a_{ij} is a nonpositive integer if $i \neq j$ (to ensure that the Serre relations (4) make sense),

(C3) $a_{ij} = 0$ implies $a_{ji} = 0$ (because of the Serre relations $(\operatorname{ad} e_i)^{1-a_{ij}} e_j = 0$ and $(\operatorname{ad} e_j)^{1-a_{ji}} e_i = 0$).

The resulting Lie algebra $\mathfrak{g}_A$ is the *Kac–Moody algebra* associated to A (or rather, its derived Lie algebra, see Chapter 3 for more details).

Another, maybe more illuminating, way to introduce Kac–Moody algebras, is to ask the following question: which Lie algebras $\mathfrak{g}$ can one obtain by keeping the following fundamental properties of finite-dimensional simple Lie algebras:

(KM1) *Generation by ℓ linearly independent copies* $\mathfrak{g}_{(i)} \subseteq \mathfrak{g}$ *of* $\mathfrak{sl}_2(\mathbb{C})$ $(i \in I := \{1, \ldots, \ell\})$. Let us write $\mathfrak{g}_{(i)} = \mathbb{C} f_i \oplus \mathbb{C}\alpha_i^\vee \oplus \mathbb{C} e_i$, where e_i, f_i and $\alpha_i^\vee := [f_i, e_i]$ are respectively identified with the matrices $\left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$, $\left(\begin{smallmatrix} 0 & 0 \\ -1 & 0 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ of $\mathfrak{sl}_2(\mathbb{C})$. In other words, we have the relations

$$[\alpha_i^\vee, e_i] = 2e_i, \quad [\alpha_i^\vee, f_i] = -2f_i \quad \text{and} \quad [f_i, e_i] = \alpha_i^\vee \quad \text{for all } i \in I. \tag{5}$$

(KM2) ad-*diagonalisability of the "Cartan subalgebra"* $\mathfrak{h} := \sum_{i \in I} \mathbb{C}\alpha_i^\vee$, with the generators e_i, f_i as eigenvectors. Thus $[\alpha_i^\vee, e_j] = a_{ij} e_j$ and $[\alpha_i^\vee, f_j] = b_{ij} f_j$ for some $a_{ij}, b_{ij} \in \mathbb{C}$. Note that, since the elements of $\mathfrak{h}$ are simultaneously diagonalisable, they commute:

$$[\alpha_i^\vee, \alpha_j^\vee] = 0 \quad \text{for all } i, j \in I. \tag{6}$$

In particular, $0 = [\alpha_i^\vee, [f_j, e_j]] = (b_{ij} + a_{ij})\alpha_j^\vee$, that is, $b_{ij} = -a_{ij}$ for all $i, j \in I$. Together with (5), this implies that for all $i, j \in I$,

$$[\alpha_i^\vee, e_j] = a_{ij} e_j \quad \text{and} \quad [\alpha_i^\vee, f_j] = -a_{ij} f_j \tag{7}$$

for some $a_{ij} \in \mathbb{C}$ with $a_{ij} = 2$ if $i = j$. Write $\mathfrak{g} = \bigoplus_{\alpha \in \mathfrak{h}^*} \mathfrak{g}_\alpha$ for the ad($\mathfrak{h}$)-eigenspace decomposition of $\mathfrak{g}$, and let

$$\Delta := \{\alpha \in \mathfrak{h}^* \setminus \{0\} \mid \mathfrak{g}_\alpha \neq 0\}$$

be the corresponding set of *roots*. Thus, if $\alpha_j \in \mathfrak{h}^*$ is defined by $\langle \alpha_j, \alpha_i^\vee \rangle := a_{ij}$ for all $i \in I$, we have $e_j \in \mathfrak{g}_{\alpha_j}$ and $f_j \in \mathfrak{g}_{-\alpha_j}$. In particular, $\pm \alpha_j \in \Delta$. Moreover, since $\mathfrak{g}$ is generated by the e_j, f_j ($j \in I$), every other root $\alpha \in \Delta$ belongs to $Q := \sum_{i \in I} \mathbb{Z}\alpha_i$ (this follows by induction using the fact that if $x \in \mathfrak{g}_\alpha$, then $[e_i, x] \in \mathfrak{g}_{\alpha + \alpha_i}$ and $[f_i, x] \in \mathfrak{g}_{\alpha - \alpha_i}$).

(KM3) *The set* $\Pi := \{\alpha_i \mid i \in I\}$ *is a root basis*[2] *of* Δ. This means that Π is a linearly independent subset of $\mathfrak{h}^*$ and that every root $\alpha \in \Delta$ is of the form $\alpha = \epsilon \sum_{i \in I} n_i \alpha_i$ for some $\epsilon \in \{\pm 1\}$ and some $n_i \in \mathbb{N}$.

In particular, $\alpha_i - \alpha_j \notin \Delta \cup \{0\}$ for $i \neq j$, so that $[e_i, f_j] = 0$ if $i \neq j$. Together with (5), this implies that

$$[f_i, e_j] = \delta_{ij} \alpha_i^\vee \quad \text{for all } i, j \in I. \tag{8}$$

(KM4) *Integrability of* $\mathfrak{g}$. This means that for each $i \in I$, the operators $\text{ad } e_i$, $\text{ad } f_i$ of $\text{End}(\mathfrak{g})$ are *locally nilpotent*: for each $x \in \mathfrak{g}$, there is some $N \in \mathbb{N}$ such that $(\text{ad } e_i)^N x = 0$ (resp. $(\text{ad } f_i)^N x = 0$). In other words, the exponential $\exp \text{ad } e_i := \sum_{s \geq 0} \frac{(\text{ad } e_i)^s}{s!}$ yields a finite sum in $\mathfrak{g}$ whenever it is evaluated on some $x \in \mathfrak{g}$, and hence defines an automorphism of $\mathfrak{g}$ (and similarly for $\exp \text{ad } f_i$). As a consequence, the adjoint action of each copy $\mathfrak{g}_{(i)}$ of $\mathfrak{sl}_2(\mathbb{C})$ on $\mathfrak{g}$ can be "integrated" to a group action $\text{SL}_2(\mathbb{C}) \to \text{Aut}(\mathfrak{g})$ (whence the terminology): this condition thus ensures that one can (at least locally) "integrate" the Lie algebra $\mathfrak{g}$ to a group G (see the next section).

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[2] When the matrix $A := (a_{ij})_{i,j \in I}$ is singular, this condition is actually too restrictive with the definition of roots we gave; this situation is discussed in detail at the beginning of Chapter 3.

Let $i, j \in I$ with $i \neq j$, and let $N \geq 1$ be minimal such that $(\mathrm{ad}\, e_i)^N e_j = 0$. An easy induction on $m \geq 1$ using (7) and (8) yields that

$$(\mathrm{ad}\, f_i)(\mathrm{ad}\, e_i)^m e_j = m(m - 1 + a_{ij}) \cdot (\mathrm{ad}\, e_i)^{m-1} e_j.$$

Hence $0 = N(N - 1 + a_{ij}) \cdot (\mathrm{ad}\, e_i)^{N-1} e_j$, so that $a_{ij} = 1 - N \in -\mathbb{N}$. In particular, $A := (a_{ij})_{i,j \in I}$ is a GCM. Moreover, the same argument with f_i yields

$$(\mathrm{ad}\, e_i)^{1-a_{ij}} e_j = 0 \quad \text{and} \quad (\mathrm{ad}\, f_i)^{1-a_{ij}} f_j = 0 \tag{9}$$

for all $i, j \in I$ with $i \neq j$.

Note that the relations (6), (7) and (8) sum up to the relations (3), while (9) coincides with the Serre relations (4). Hence if $\mathfrak{g}$ is a Lie algebra satisfying (KM1)–(KM4), then its associated matrix A is a GCM and $\mathfrak{g}$ is a quotient of $\mathfrak{g}_A$. In other words, the Kac–Moody algebras $\mathfrak{g}_A$ are the "most general" Lie algebras satisfying (KM1)–(KM4). In fact, by an important theorem of Gabber and Kac, $\mathfrak{g}_A$ is simple (modulo center contained in $\mathfrak{h}$) in many cases (and conjecturally in all cases); in particular, in such cases, $\mathfrak{g} \cong \mathfrak{g}_A$, and hence $\mathfrak{g}_A$ is characterised by (KM1)–(KM4).

The Kac–Moody algebra $\mathfrak{g}_A$ associated to a GCM A is infinite-dimensional as soon as A is not a Cartan matrix. The *root spaces* $\mathfrak{g}_\alpha$ ($\alpha \in \Delta$) remain, however, of finite dimension. Certain roots $\alpha \in \Delta$, such as the *simple roots* α_i ($i \in I$), behave exactly as the roots of a simple finite-dimensional Lie algebra. In particular, their associated root space $\mathfrak{g}_\alpha$ has the following properties:

(RR1) dim $\mathfrak{g}_\alpha = 1$.

(RR2) ad $x \in \mathrm{End}(\mathfrak{g}_A)$ is locally nilpotent for each $x \in \mathfrak{g}_\alpha$.

These roots are called *real*, and their set is denoted Δ^{re}. The key novelty of infinite-dimensional Kac–Moody algebras is the apparition of roots with a totally different behaviour, which one calls *imaginary roots* (their set is $\Delta^{\mathrm{im}} := \Delta \setminus \Delta^{\mathrm{re}}$). This new behaviour is illustrated by the following properties of the root space $\mathfrak{g}_\beta$ of an imaginary root $\beta \in \Delta^{\mathrm{im}}$:

(IR1) dim $\mathfrak{g}_\beta$ is, in general, bigger than 1.

(IR2) ad $x \in \mathrm{End}(\mathfrak{g}_A)$ is *not* locally nilpotent for any nonzero $x \in \mathfrak{g}_\beta$.

Example 3. Cartan matrices are of course particular cases of GCM (specifically, they are the GCM of the form $A = DB$ for some diagonal matrix D and some symmetric positive definite matrix B), and hence the simple finite-dimensional Lie algebras, such as $\mathfrak{sl}_n(\mathbb{C})$, are Kac–Moody algebras, of so-called *finite type*.

The next type of GCM, by increasing order of complexity, are the GCM A of *affine type* (specifically, they are the GCM of the form $A = DB$ for some diagonal matrix D and some symmetric positive semi-definite matrix B of corank

1). In that case $\mathfrak{g}_A$ is infinite-dimensional, but its size remains "controlled", in the following sense. Associate to each root $\alpha = \epsilon \sum_{i \in I} n_i \alpha_i$ ($\epsilon \in \{\pm 1\}$, $n_i \in \mathbb{N}$) its *height* $\mathrm{ht}(\alpha) := \epsilon \sum_{i \in I} n_i \in \mathbb{Z}$. Since all $\mathfrak{g}_\alpha$ are finite-dimensional, we obtain a function

$$\mathrm{growth}_A \colon \mathbb{N} \to \mathbb{N}, \quad \mathrm{growth}_A(n) := \dim \bigoplus_{|\mathrm{ht}(\alpha)| \leq n} \mathfrak{g}_\alpha = \sum_{|\mathrm{ht}(\alpha)| \leq n} \dim \mathfrak{g}_\alpha.$$

If A is of affine type, then $\mathfrak{g}_A$ is of *polynomial growth*, in the sense that $\mathrm{growth}_A(n)$ grows as a polynomial in n for $n \to \infty$. As a result, Kac–Moody algebras of affine type are still well understood; in particular, they possess explicit realisations as matrix algebras over the ring $\mathbb{C}[t, t^{-1}]$ of Laurent polynomials. For instance, if $A = \left(\begin{smallmatrix} 2 & -2 \\ -2 & 2 \end{smallmatrix} \right)$, then $\mathfrak{g}_A$ is isomorphic to (a one-dimensional central extension of) $\mathfrak{sl}_2(\mathbb{C}[t, t^{-1}])$.

In all other cases (hence in the vast majority of cases), A is said to be of *indefinite type*. The Kac–Moody algebra $\mathfrak{g}_A$ is then of *exponential growth*, in the sense that $\mathrm{growth}_A(n)$ grows as an exponential in n for $n \to \infty$. Such Kac–Moody algebras remain mysterious to a large extent; in particular, unlike the affine case, one does not currently know of any "concrete realisation" of any such $\mathfrak{g}_A$. ☐

3 Kac–Moody groups

Let $\mathfrak{g}_A = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta} \mathfrak{g}_\alpha$ be a Kac–Moody algebra (the notations being as in the previous section). In trying to reproduce the classical (i.e. finite-dimensional) Lie theory in our infinite-dimensional setting, the next step is to ask whether one can construct a group "attached" (in any reasonable sense) to $\mathfrak{g}_A$. In other words:

> How can one construct a group G_A deserving the name of "Kac–Moody group of type A"?

The answer to this question very much depends on the category of groups one wants to consider: for instance, to the finite-dimensional simple Lie algebra $\mathfrak{sl}_n(\mathbb{C})$, one can attach the group $\mathrm{SL}_n(\mathbb{C})$, which is not just an abstract group, but also a topological group, and even a Lie group. With a more algebraic perspective, one can also attach to $\mathfrak{sl}_n(\mathbb{C})$ the *group functor* SL_n associating to each (commutative, unital) ring k the group

$$\mathrm{SL}_n(k) := \{B \in \mathrm{Mat}(n \times n, k) \mid \det B = 1\};$$

this group functor even has the structure of an *affine group scheme* (see Appendix A for a short introduction to these notions).

As we will see in the third part of this book, analogues for each of the above mentioned group structures associated to $\mathfrak{sl}_n(\mathbb{C})$ can be obtained in the setting of

general Kac–Moody algebras, with the exception of a smooth Lie group structure, which remains at present elusive. On the other hand, new structures that are specific to the infinite-dimensional setting also arise (for instance, topological group structures that are only non-discrete when the group is infinite-dimensional).

Depending on the targeted category of groups, the required amount of effort to construct a "Kac–Moody group" may greatly vary. For instance, an analogue of $SL_n(\mathbb{C})$ (or rather, $PSL_n(\mathbb{C})$) as an abstract group for an arbitrary Kac–Moody algebra $\mathfrak{g}_A$ can be defined straightaway. Indeed, keeping the notations of Examples 1 and 2, we recall that $SL_n(\mathbb{C})$ is generated by its *root groups*

$$U_{\alpha_{ij}} := \exp(\mathfrak{g}_{\alpha_{ij}}) = \{\mathrm{Id} + rE_{ij} \mid r \in \mathbb{C}\} \quad \text{for } i \neq j$$

exponentiating the root spaces of $\mathfrak{sl}_n(\mathbb{C})$. To give a sense to exponentiation and generation by subgroups for a general Kac–Moody algebra $\mathfrak{g}_A$, one naturally considers the "ambient space" $\mathrm{Aut}(\mathfrak{g}_A)$. Since by (KM4) (and, more specifically, (RR2)), the exponentials $\exp \mathrm{ad}\, x = \sum_{s \geq 0} \frac{(\mathrm{ad}\, x)^s}{s!}$ define elements of $\mathrm{Aut}(\mathfrak{g}_A)$ for $x \in \mathfrak{g}_\alpha$ whenever $\alpha \in \Delta$ is a real root, this suggests to define the group

$$G_A := \langle \exp \mathrm{ad}\, x \mid x \in \mathfrak{g}_\alpha,\ \alpha \in \Delta^{\mathrm{re}} \rangle \subseteq \mathrm{Aut}(\mathfrak{g}_A),$$

which one might call an "adjoint complex Kac–Moody group of type A". For instance, if $\mathfrak{g}_A = \mathfrak{sl}_n(\mathbb{C})$, then $\Delta^{\mathrm{re}} = \Delta$ and G_A is the image of $SL_n(\mathbb{C})$ in $\mathrm{Aut}(\mathfrak{g}_A)$, that is, $G_A \cong PSL_n(\mathbb{C})$. If $A = \left(\begin{smallmatrix} 2 & -2 \\ -2 & 2 \end{smallmatrix}\right)$ as in Example 3, one can check that $G_A \cong PSL_2(\mathbb{C}[t, t^{-1}])$.

The group G_A is an example of a *minimal* Kac–Moody group, in the sense that it is constructed by only exponentiating the *real* root spaces of $\mathfrak{g}_A$. If one also takes into account *imaginary* root spaces, one typically obtains a certain completion of a minimal Kac–Moody group, called a *maximal* Kac–Moody group.

Example 4. Let $A = \left(\begin{smallmatrix} 2 & -2 \\ -2 & 2 \end{smallmatrix}\right)$, so that $\mathfrak{g}_A = \mathfrak{sl}_2(\mathbb{C}[t, t^{-1}])$ (up to a central extension by $\mathbb{C}$). One shows that $\Delta^{\mathrm{im}} = \mathbb{Z}_{\neq 0}\delta$ for some imaginary root δ. Moreover, the imaginary root space $\mathfrak{g}_{n\delta}$ of $\mathfrak{g}_A$ is spanned by the diagonal matrix $x_n := \left(\begin{smallmatrix} t^n & 0 \\ 0 & -t^n \end{smallmatrix}\right)$. Note that $\mathrm{ad}\, x_n$ is not locally nilpotent (for instance, $(\mathrm{ad}\, x_n)^s \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 0 & 2^s t^{ns} \\ 0 & 0 \end{smallmatrix}\right) \neq 0$ for all $s \in \mathbb{N}$), as indicated by (IR2). In other words, if one wishes to exponentiate x_n, one has to allow formal power series in t: the exponential $\exp x_n$ makes sense in the *maximal Kac–Moody group* $\widehat{G}_A := SL_2(\mathbb{C}((t)))$ for each $n \in \mathbb{N}$, where $\mathbb{C}((t))$ denotes the field of complex formal Laurent series. $\qquad\square$

While the minimal versions of Kac–Moody groups are easier to define, they are also usually harder to study. On the other hand, the various Kac–Moody groups one can construct come in very different flavours. For instance, the minimal Kac–Moody group G_A defined above can be turned into a connected Hausdorff topological group; maximal Kac–Moody groups defined over finite fields,

on the other hand, are naturally (non-discrete if A is not of finite type) totally disconnected locally compact groups (think of $\mathrm{SL}_n(\mathbb{C}[t, t^{-1}])$ versus $\mathrm{SL}_n(\mathbb{F}_q((t)))$ for $\mathbb{F}_q$ a finite field of order q).

In the third part of this book, we explore in detail the following questions:

- What are the possible (currently known) constructions of Kac–Moody groups, and how much additional structure can they be equipped with?

- How do these constructions relate to one another; is there a unique "good" definition of Kac–Moody group?

Along the way, we will encounter one of the most powerful tools to study Kac–Moody groups: buildings. These are certain simplicial complexes on which Kac–Moody groups act nicely (an introduction to buildings and groups acting on them is given in Appendix B). They yield a geometric interpretation of many important properties of Kac–Moody groups, thereby adding to this algebraic subject a nice geometric flavour.

4 Structure of the book and guide to the reader

The purpose of the first part of the book, on the classical Lie theory, is to set the scene. It introduces many of the concepts that will appear in the more general setting of Kac–Moody algebras, thus providing some motivation and basic examples for these concepts. We also included in Part I some proofs, whenever they provided some intuition for the kind of arguments involved in the study of Kac–Moody algebras. Part I is thus helpful in smoothening the path to Kac–Moody algebras; however, logically speaking, it is independent of the rest of the book, and the impatient reader may safely jump to the second and third parts of the book.

The second part of this book serves as an introduction to Kac–Moody algebras. There are several good references on the topic, including the standard book [Kac90] by V. Kac, from which most of the material from Part II is taken. Here, we chose a minimal, but nevertheless self-contained path to Kac–Moody groups, trying to provide some extra intuition whenever we felt it necessary, and to smoothen the occasional rough spots of [Kac90].

Section 3.7 in Chapter 3 and most of Chapter 5 could be omitted as far as the general theory of Kac–Moody groups is concerned; however, they are necessary to understand Kac–Moody algebras and groups of *affine type* in more details. Since these are the only available source of "concrete" examples of Kac–Moody algebras and groups (besides the finite-dimensional ones), it is nevertheless worthwhile to spend some time on exploring them further.

The heart of this book is of course its third part, on the construction and basic properties of Kac–Moody groups. We start Chapter 7 by following the most obvious path to attaching a group to a given Kac–Moody algebra. The resulting

group G, although certainly deserving the name of "Kac–Moody group", does not, however, give a totally satisfactory answer to the problem of attaching groups to Kac–Moody algebras. We express four natural concerns about the construction of G (namely, the problematics (P1)–(P4) in §7.1.2). The rest of Chapter 7 is then devoted to answering these concerns.

The structure of Chapter 8 is similar: we start by expressing two additional concerns (the problematics (P5)–(P6) in §8.1) about the objects introduced in Chapter 7 (the *minimal* Kac–Moody groups), and devote the rest of Chapter 8 to answering these concerns. This leads to the construction of *maximal* Kac–Moody groups, obtained as some completions of the minimal ones.

The progression of Chapters 7 and 8 is essentially linear. These chapters provide constructions of Kac–Moody groups at various levels of generality and from various perspectives. The reader should feel free to evaluate for him-/herself which of the concerns (P1)–(P6) are relevant to his/her needs or interests, and decide accordingly how far to go in exploring the proposed answers to these concerns (it should be clear from the beginning of each section which of the problematics (P1)–(P6) that section addresses). Some sections are also marked by an asterisk, indicating that they are not logically required to study the subsequent sections (without asterisk). Such sections essentially fall into two (related) categories: first, the explicit constructions of affine Kac–Moody algebras and groups, and second, results related to *Kac–Moody root data* (see §7.3.1). The latter topic, a useful (especially in the affine case) but more technical aspect in the construction of Kac–Moody groups, could even be entirely avoided on a first reading without hindering the comprehension of the subsequent sections. As the vocabulary of Kac–Moody root data is nevertheless used throughout Chapters 7 and 8, the reader is then refered to Remark 7.17 at the end of §7.3.1, which indicates the necessary translations to be made.

Chapter 9 consists of a few short sections reviewing some selected important questions or research directions pertaining to Kac–Moody groups. We could have added many more sections reflecting other important aspects of Kac–Moody theory: the proposed selection is thus very far from being an exhaustive overview of the theory. These sections can be read independently of one another.

We conclude the book with two appendices, offering short introductions to the topic of affine group schemes (Appendix A), and to the topic of buildings and groups acting on them (Appendix B). Section A.1 recalls the basic vocabulary of categories and functors, needed from the beginning of Part III. The rest of Appendix A comes into play later on, around §8.5. The content of Appendix B becomes important in §7.4.6 (as well as in Chapter 8), mainly to provide some geometric intuition. Suggestions of appropriate timings to go through each appendix are also included within the text.

5 Conventions

Throughout the book, we denote by $\mathbb{N} = \{0, 1, \ldots\}$ the set of nonnegative integers and by $\mathbb{N}^* = \{1, 2, \ldots\}$ the set of positive integers. As usual, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$ respectively denote the sets of integers, rational numbers, real numbers and complex numbers. We further set $\mathbb{Z}^* := \mathbb{Z} \setminus \{0\}$.

By a *ring* we always mean a commutative, unital, associative $\mathbb{Z}$-algebra. For a ring k, we write $k^\times$ for the set of its invertible elements.

Finally, if $x \in \mathbb{C}$ and $n \in \mathbb{N}$, we let $\binom{x}{n}$ denote the binomial coefficient

$$\binom{x}{n} := \frac{x(x-1)\cdots(x-n+1)}{n!} \quad \text{if } n > 0 \quad \text{and} \quad \binom{x}{0} := 1.$$