Contents

1 Estimates for matrix operators 1
1.0 Introduction 1
1.0.1 Description of results 1
1.0.2 Outline of the proof of the main result 5
1.1 Estimates for systems of ordinary differential operators on a semi-axis 12
1.1.1 Some assumptions and notation 12
1.1.2 Transformation of the basic inequality 13
1.1.3 The simplest lower bound of the constant Λ 14
1.1.4 On solutions of the system $\mathscr{P}_+(-\mathrm{i}\,d/dt)\,I\varphi = 0$ 15
1.1.5 Properties of the matrix $T(\tau)$ 19
1.1.6 Integral representation for $\dot{S}(-\mathrm{i}\,d/dt)\,\psi$ 21
1.1.7 Properties of the matrix $G(\tau)$ 24
1.1.8 A quadratic functional 27
1.1.9 Necessary and sufficient conditions for the validity of inequality (1.1.1) 29
1.1.10 On condition 4 of Theorem 1.1.19 32
1.1.11 Matrix $G(\tau)$ for estimates with a "large" number of boundary operators 35
1.1.12 Explicit representations of the matrix $G(\tau)$ 37
1.1.13 Estimates for vector functions satisfying homogeneous boundary conditions 38
1.1.14 Estimates for vector functions without boundary conditions . . 39
1.2 Estimates in a half-space. Necessary and sufficient conditions 40
1.2.1 Basic assumptions and notation 41
1.2.2 Theorems on necessary and sufficient conditions for the validity of the estimates in a half-space 43
1.2.3 Matrix $G(\xi;\tau)$ and its properties 45
1.2.4 The case of a single boundary operator 48
1.2.5 The case of a polynomial $\mathscr{P}(\xi;\tau)$ with roots in the half-plane $\operatorname{Im}\zeta \leqslant 0$ 50
1.2.6 Estimates of the types (1.2.1), (1.2.12), (1.2.13) in the norms $\|\cdot\|_{\nu}$ and $\langle\!\langle\cdot\rangle\!\rangle_{\mu}$ 50
1.2.7 The case, where the lower-order terms have no influence . . . 53
1.3 Estimates in a half-space. Sufficient conditions 57
1.3.1 Sufficient condition for the validity of the estimate (1.3.1) . . . 58
1.3.2 The case $\mathfrak{M}(\xi) = \mathscr{T}_+^{-1/2}(\xi)$ 60
1.3.3 The case of the diagonal matrix $\mathfrak{M}(\xi)$ 65
1.3.4 Sufficient conditions for the validity of the estimate (1.3.21) . . 66

1.4 Examples . . . 69
1.4.1 Generalized-homogeneous quasielliptic systems . . . 69
1.4.2 The Lamé system of the static elasticity theory . . . 72
1.4.3 The Cauchy–Riemann system . . . 74
1.4.4 The stationary linearized Navier–Stokes system . . . 76
1.4.5 Hyperbolic systems . . . 77
1.4.6 Operators of first order in the variable t. Scalar case . . . 80
1.4.7 An example of a second-order operator w.r.t. t . . . 82
1.5 On well-posed boundary value problems in a half-space . . . 84
1.6 Notes . . . 89

2 Boundary estimates for differential operators . . . 93
2.0 Introduction . . . 93
2.0.1 Description of results . . . 93
2.0.2 Outline of the proof of the main result . . . 95
2.1 Estimates for ordinary differential operators on the semi-axis . . . 99
2.1.1 A lemma on polynomials . . . 100
2.1.2 A variational problem in finite-dimensional space . . . 106
2.1.3 Reduction of the estimate for ordinary differential operators on the semi-axis to a variational problem in a finite-dimensional space . . . 110
2.1.4 Two properties of the matrix $\mathfrak{B}$ . . . 114
2.1.5 An estimate without boundary operators in the right-hand side . . . 115
2.1.6 Necessary and sufficient conditions for the validity of inequality (2.1.1) . . . 117
2.1.7 Estimates for functions satisfying homogeneous boundary conditions . . . 119
2.2 Estimates in a half-space. Necessary and sufficient conditions . . . 122
2.2.1 Theorems on necessary and sufficient conditions for the validity of the estimates in a half-space . . . 123
2.2.2 Corollaries . . . 125
2.2.3 The case when the lower-order terms play no role . . . 128
2.2.4 An example of estimate for operators of first order with respect to t . . . 130
2.3 Description of the trace space . . . 133
2.3.1 Preliminary results . . . 133
2.3.2 Embedding and extensions theorems . . . 137
2.3.3 On the extension of functions from $\mathscr{H}(\mathbb{R}^n)$ to $\mathscr{H}(\mathbb{R}^n_+)$ . . . 141
2.4 Notes . . . 144

3 Dominance of differential operators . . . 145
3.0 Introduction . . . 145
3.0.1 Description of results . . . 145
3.0.2 Remarks on the method of proving the main result . . . 147

3.1 Estimates for ordinary differential operators on the semi-axis 149
3.1.1 A variational problem in a finite-dimensional space 150
3.1.2 The simplest lower bound for the constant Λ 153
3.1.3 Reduction of the estimates for ordinary differential operators on the semi-axis to variational problems in a finite-dimensional space 154
3.1.4 Necessary and sufficient conditions for the validity of inequalities (3.1.1) and (3.1.1′) 158
3.1.5 Inequalities for functions without boundary conditions 161
3.2 Estimates in a half-space. Necessary and sufficient conditions 162
3.2.1 Necessary and sufficient conditions for the validity of the estimates (3.0.1), (3.0.2), and (3.0.1′) 162
3.2.2 On the minimal number and algebraic properties of the boundary operators; formulas for $\beta_\alpha(\xi;\eta)$ 165
3.2.3 Estimates for polynomials whose τ-roots lie in the lower complex half-plane 168
3.3 Examples 169
3.3.1 The theorem of N. Aronszajn on necessary and sufficient conditions for the coercivity of a system of operators 169
3.3.2 The case $m = 1$, $N = N(\xi)$ in Theorems 3.2.2, 3.2.3, and 3.2.4 170
3.3.3 Examples of estimates for operators of first order with respect to t 174
3.4 Notes 177

4 Estimates for a maximal operator 179
4.0 Introduction 179
4.1 Preliminary results 181
4.1.1 Results concerning the estimate (4.0.1) 181
4.1.2 Results concerning the estimate (4.0.2) 185
4.2 Quasielliptic polynomials 187
4.2.1 Polynomials with a generalized-homogeneous principal part . . 187
4.2.2 The estimate (4.2.16) for quasielliptic polynomials of type $l \geqslant 1$ 190
4.2.3 The estimate (4.2.19) for quasielliptic polynomials of type $l \geqslant 1$ 192
4.3 Homogeneous polynomials with simple roots 195
4.3.1 Asymptotic representations of the τ-roots of the polynomial $H_+(\xi;\tau)$ as $|\xi| \to \infty$ 196
4.3.2 Necessary and sufficient conditions for the validity of the estimate (4.2.16) 197
4.3.3 Necessary and sufficient conditions for the validity of the estimate (4.2.19) 201
4.4 Some classes of nonhomogeneous polynomials with simple roots . . . 204
4.4.1 A formula for the function $\Lambda(\xi)$ 205
4.4.2 Asymptotic representations as $N \to \infty$ for the τ-roots $\zeta_j(\xi)$ of the polynomial $H_+(\xi;\tau)$ 205

4.4.3 An asymptotic representation of the function $\Lambda(\xi)$ as $N \to \infty$ for polynomials P with the real τ-roots . . . 208
4.4.4 Necessary and sufficient conditions for the validity of the estimates (4.0.1), (4.0.2) for a polynomial P with real τ-roots . 210
4.4.5 An asymptotic representation of the function $\Lambda(\xi)$ as $N \to \infty$ for a polynomial P with the τ-roots lying in the half-plane $\operatorname{Im}\zeta < 0$ . . . 211
4.4.6 Necessary and sufficient conditions for the validity of the estimates (4.0.1), (4.0.2) for a polynomial P with the τ-roots lying in the half-plane $\operatorname{Im}\zeta < 0$ . . . 213
4.4.7 An asymptotic representation of the function $\Lambda(\xi)$ as $N \to \infty$ for a polynomial P with the τ-roots lying in the half-plane $\operatorname{Im}\zeta > 0$ . . . 214
4.4.8 Necessary and sufficient conditions for the validity of the estimates (4.0.1), (4.0.2) for a polynomial P with the τ-roots lying in the half-plane $\operatorname{Im}\zeta > 0$ . . . 215
4.5 Second-order polynomials of τ . . . 217
4.5.1 Preliminary results . . . 217
4.5.2 The case $p_1(\xi) \equiv 0$ . . . 219
4.5.3 The case $\operatorname{Im} p_k(\xi) \equiv 0$ $(k = 0, 1, 2)$ . . . 221
4.5.4 The estimate (4.2.16) in the case $\operatorname{Re} p_1(\xi) \equiv 0$, $\operatorname{Im} p_k(\xi) \equiv 0$ $(k = 0, 2)$ . . . 222
4.6 Space of traces of functions for the maximal operator . . . 226
4.6.1 The maximal operator as closure of its restriction on the set of functions infinitely differentiable up to the boundary . . . 227
4.6.2 Description of the "trace space" . . . 229
4.7 Notes . . . 233

Notation . . . 235

Bibliography . . . 239

Index . . . 243