Preface

These notes deal with spaces $S^r_{p,q}A(\mathbb{R}^n)$ of Besov–Sobolev type with dominating mixed smoothness, where $A \in \{B, F\}$, $r \in \mathbb{R}$, $0 < p \leq \infty$, $0 < q \leq \infty$, and their counterparts $S^r_{p,q}A[\Omega, \varkappa]$ in arbitrary bounded domains Ω in $\mathbb{R}^n$. Here $\varkappa$ indicates a weight of type $\operatorname{dist}(x, \partial\Omega)^\varkappa$, $x \in \Omega$.

Our motivation comes from mathematical biology, where the study of the filigree structure of the beautiful patterns created by tiny animals (several types of bacteria, amoebae, etc.) became a central topic of mathematical biology; see [Mur93, Mur03, Per15]. The underlying mathematics is governed by a large variety of so-called Keller–Segel equations. We dealt with some of them in [T17] in the framework of the isotropic inhomogeneous spaces $A^s_{p,q}(\mathbb{R}^n)$, $A \in \{B, F\}$, covering in particular (fractional) Sobolev spaces $H^s_p(\mathbb{R}^n) = F^s_{p,2}(\mathbb{R}^n)$ and Besov spaces $B^s_{p,q}(\mathbb{R}^n)$, concentrating on existence and uniqueness assertions of the initial value problems for these nonlinear parabolic differential equations. Somewhat outside the main body of [T17], in [T17, Sect. 5.6] we discussed numerical aspects, suggesting *Faber devices* being subspaces of suitable spaces $S^r_{p,q}A(\mathbb{R}^n)$ with dominating mixed smoothness. But a closer examination of this proposal shows that subspaces of $S^r_{p,q}A(\mathbb{R}^n)$ and arbitrary bounded domains Ω in $\mathbb{R}^n$ do not fit together very well (in contrast to isotropic spaces $A^s_{p,q}(\mathbb{R}^n)$ and $A^s_{p,q}(\Omega)$). The only exceptions seem to be cubes and rectangles with sides parallel to already fixed coordinate axes, their (global) fibre-preserving diffeomorphic images and, at best, a finite union of them. However, we return to this point in Section 2.5.1. In any case it might be better to introduce related weighted spaces $S^r_{p,q}A[\Omega, \varkappa]$ on arbitrary bounded domains Ω in $\mathbb{R}^n$ intrinsically. This will be done in Chapter 2 of these notes, based on Whitney decompositions of Ω into cubes (with sides parallel to the axes of a fixed system of Euclidean coordinates). But this requires knowledge of some specific properties of the related spaces $S^r_{p,q}A(\mathbb{R}^n)$ which are not available in the literature so far, for example homogeneity at the small, suitable pointwise multipliers etc. It is one aim of Chapter 1 to deal with corresponding properties, but we hope that Chapter 1 is also of self-contained interest. It complements the theory of the spaces $S^r_{p,q}A(\mathbb{R}^n)$ and $S^r_{p,q}A(Q)$ with $Q = (0,1)^n$ as it may be found in the surveys [ScS04, Schm07, Vyb06] and in the relevant chapters of the books [ST87, T10]. The corresponding theory of the closely related periodic spaces with dominating mixed smoothness $S^r_{p,q}A(\mathbb{T}^n)$ on the n-torus $\mathbb{T}^n$ may be found in [Tem93, Tem03] and also in the recent survey [DTU16] with a comprehensive bibliography of more than 400 items.

We fix our use of $\sim$ (equivalence) as follows. Let I be an arbitrary index set. Then

$$a_i \sim b_i \quad \text{for } i \in I \text{ (equivalence)} \tag{P.1}$$

for two sets of positive numbers $\{a_i : i \in I\}$ and $\{b_i : i \in I\}$ means that there are two positive numbers c_1 and c_2 such that

$$c_1 a_i \le b_i \le c_2 a_i \quad \text{for all } \ i \in I. \tag{P.2}$$