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Todd Fisher
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Hyperbolic Flows

European Mathematical Society

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Anatole Katok
in memoriam

(Author: Konrad Jacobs. Source: Photo collection of the Mathematical Research Institute of Oberwolfach, Germany.)

Acknowledgments

On Thu, 15 May 2014, Todd Fisher wrote,

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Boris,
I hope things are going well for you.  I have an undergraduate student working
on a problem about hyperbolic flows.  I am having a hard time coming up with a
survey or reference for the background material.  Mostly, I find books and
surveys on hyperbolic maps and then in the papers they apply these to flows
without specific references.  It would help his understanding if there was such
a reference.  I was hoping you might know of one.  If you do can you send me
some places to look.
Best, Todd
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On Thu, May 15, 2014 and Fri, 16 May 2014 Boris Hasselblatt wrote,

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Dear Todd,
Greetings from Somerville.  I hope that you are well.
I am here for a few days between a 6-month stint in Marseille and a month in
Tokyo.  3 cheers for sabbaticals.
...
“Introduction to the Modern Theory of Dynamical Systems” is largely guilty of
the sins you mention, but Tolya and I have long been working on a successor, and
some of the intent is to have fewer transgressions of this sort (also in ergodic
theory).  I could send the current PDF...
...
While we do more with flows (with respect to hyperbolicity but also ergodic
theory) than in “Introduction to the Modern Theory of Dynamical Systems,” there
may be a place for a survey that makes a point of it.  It might be interesting
to write one, and I am game.
...
Best regards
Boris
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Thus emerged the modest germ of a plan to collect in one place the core of hyperbolic dynamics with continuous time. That summer, Hasselblatt became Associate Provost at Tufts, which significantly impacted this project. Modest progress occurred until Fisher was able to spend a significant amount of time on the book in the first half of 2016 during a sabbatical. In 2016, Hasselblatt's assignment of Associate Provost at Tufts ended with the commitment by the Office of the Provost to fund a semester leave. Soon after, Hasselblatt was also informed that a previously deferred sabbatical was due to expire if not taken by the spring semester of 2018. Thus, a spring 2017 topics course at Tufts, spring 2018 Nachdiplom lectures in Zürich, and fall 2018 lectures in Tokyo hugely boosted this project, and it grew well beyond its initial scope by embracing a far broader view of "core," including some new results, and by encompassing a concise and lively introduction to important areas of current research, variously with proofs, proof outlines, proof ideas, or references to proofs.

There is much to acknowledge that has significantly helped us write this book. We owe a debt to our respective institutions for their support, and to the Simons Foundation for providing research support for the book.[1] We also want to thank

[1] Simons Collaborative Grants # 585027 and # 587001.

students from Brigham Young University, Tufts University, Brandeis University, the University of Tokyo, and the ETH Zürich for their forbearance, support, and criticism;[2] colleagues and students who commented helpfully on book drafts from afar; Manfred Einsiedler and Michael Struwe for arranging the Nachdiplom lectures at the ETH in the spring of 2018; their colleagues and Grete Einsiedler for making Zürich a home; Takashi Tsuboi and Masahiko Kanai for arranging lectures on hyperbolic flows at the Graduate School of Mathematical Sciences of the University of Tokyo in the fall of 2018; their colleagues in Tokyo, Kyoto, Nagoya, and beyond for their interest and warm hospitality; Thomas Kappeler for shepherding this project into publication; and Thomas Hintermann from the European Mathematical Society for his enthusiasm, wisdom, and support in all stages of the publication process over more than a year, right up to the start of the publication process; and Vera Spillner who, with Sylvia Fellmann and Alison Durham, took this project to completion from there with exemplary attention to quality. It seems highly appropriate and satisfying that during the last stages of writing, Boris Hasselblatt was a department colleague of Masahiko Kanai, whose work was foundational for substantial parts of the rigidity theory described near the end of the book, as well as of Shuhei Hayashi, who with his proof of the stability theorems for hyperbolic flows placed one of the crowning glories atop hyperbolic dynamics in the 20th century. Toshitake Kohno, Dean of the Graduate School of Mathematical Sciences, not only provided a most conducive environment but also access to the model collection and permission to make the photograph in Figure 5.2.6. Boris Hasselblatt is grateful for and will long remember the generous hospitality and outstanding working conditions of both the Graduate School of Mathematical Sciences at the University of Tokyo and the Forschungsinstitut für Mathematik at the ETH Zürich. Both of us thank our respective home institutions for the leaves which made this project possible. We are also grateful to the faculty writing group in the Department of Mathematics at Brigham Young University for the many suggestions and improvements they provided.

Some of this book is owed to earlier books and research articles by one or the other of us, which included text we deemed—in more or less adapted form—to be an excellent fit for this work. This implies a debt to our respective coauthors of such prior works, Anatole Katok foremost among them. Indeed, some of this text is adapted from [213] and from unfinished projects of Katok and Hasselblatt. In some cases, original research papers by others still remain the best exposition of ideas we could not omit from this book, so it will be apparent and often explicit where we followed their ideas; Bowen foremost comes to mind. And occasionally, unpublished lecture notes (such as by Lanford at the ETH) provided the most elegant proofs we know of a needed fact.

[2] In the Talmud, R. Chanina remarked, "I have learned much from my teachers, more from my colleagues, and the most from my students" (Ta'anis 7a).

In that category we particularly appreciate having the permission of Flavio Abdenur and Marcelo Viana to reproduce their proof of absolute continuity of the invariant foliations in the generality of partially hyperbolic dynamical systems (Section B.7.b). We are also grateful to the countless colleagues who generously answered questions, read drafts, and commented on the text, Clark Butler, Vaughn Climenhaga, and Daniel Thompson foremost among them: Ethan Akin, Joseph Auslander, Lennard Bakker, Aaron Brown, Keith Burns, Manfred Einsiedler, Sergio Fenley, Andrey Gogolev, Shuhei Hayashi, François Ledrappier, and Davi Obata. Several of them also encouraged us greatly by pointing out just how valuable a reference this text is, even for researchers.

Above all, we are grateful to our families for the support they provided for our obsession and absences. Mary Fisher was supportive of her husband, while he traveled often, so he could work on the book. Kathleen Hasselblatt deserves particular gratitude and recognition for dealing with the adversities of an old house and the various challenges of New England seasons while her husband lived on other continents for most of 2018—during which year some 500 pages of this book were written. We could not have done it without them.

Provo and Medford, May 2019
Todd Fisher, Boris Hasselblatt