Chapter 1
Introduction

This chapter contains a concise mathematical background. We present the basic notation and functional inequalities used in different parts of the book. Since the inequalities are used mainly for quantitative analysis, special attention is paid to computable estimates of the corresponding constants. Also, the chapter includes a literature overview and an outline of the material exposed in subsequent chapters.

1.1 Basic notation

1.1.1 Domains and operators

Throughout the book we denote domains by the letters Ω and ω. They are assumed to be open, bounded, and connected sets in the Euclidean space $\mathbb{R}^d$, where $d \in \mathbb{N}_{>0}$. Here, $\mathbb{N} := \{0, 1, \ldots\}$ is the set of natural numbers and $\mathbb{N}_{>0} := \mathbb{N} \setminus \{0\}$.

By $\mathbb{R}$ and $\mathbb{R}_{>0}$ we denote the set of real numbers and the set of positive real numbers, respectively. In some cases, it is convenient to use the extended set $\overline{\mathbb{R}}$ of real numbers, which contains $-\infty$ and $+\infty$. The vector space $\mathbb{R}^d$ is endowed with the cartesian coordinate system, so that a point $\mathbf{x}$ has coordinates $(x_1, x_2, \ldots, x_d)$. $\mathrm{B}(\mathbf{x}, \delta)$ denotes the open ball of radius δ centered at $\mathbf{x} \in \mathbb{R}^d$.

All the domains are assumed to be bounded and have Lipschitz continuous boundary (denoted $\partial\Omega$, $\partial\omega$, or Γ), which may have several nonintersecting parts (e.g., Γ_1 and Γ_2). By $\mathbf{n}$ we denote the outward unit normal to Γ. The diameter of the set Ω and its Lebesgue measure are denoted by $\operatorname{diam}\Omega$ and $|\Omega|$, respectively.

Latin letters (e.g., u, v, w) are typically used to denote scalar-valued functions. We use special (sans serif or bold) letters $\mathbf{p}, \mathbf{q}, \mathbf{y}, \boldsymbol{\eta}$ to indicate that the object is a vector or a vector-valued function. The same rule is used for matrices and tensor-valued functions (e.g., A, $\boldsymbol{\sigma}(\mathbf{x})$, $\boldsymbol{\tau}(\mathbf{x})$). All the quantities are assumed to be real-valued.

Calligraphic and capital Greek letters (e.g., $\mathcal{B}$, Λ) are used for the operators and functionals. $\mathcal{L}(X, Y)$ denotes the space of bounded linear operators acting from X to Y.

The scalar product of vectors is denoted by the dot, i.e.

$$\mathbf{p} \cdot \mathbf{q} := \sum_{i=1}^{d} p_i q_i, \quad \mathbf{p}, \mathbf{q} \in \mathbb{R}^d,$$

where the symbol := means "equals by definition". Analogously, the product of $d \times d$ matrices (or matrix-valued functions) is denoted by a colon, i.e.,

$$\boldsymbol{\varepsilon} : \boldsymbol{\sigma} := \sum_{i,j=1}^{d} \varepsilon_{ij}\sigma_{ij}.$$

Norms of vectors and matrices are associated with the respective scalar products, namely,

$$|\mathbf{q}| := (\mathbf{q} \cdot \mathbf{q})^{1/2}, \qquad |\boldsymbol{\sigma}|^2 := (\boldsymbol{\sigma} : \boldsymbol{\sigma})^{1/2}.$$

Note that $|\boldsymbol{\sigma}|$ is called the Frobenius norm of $\boldsymbol{\sigma}$. The tensor product of two vectors $\mathbf{a}$ and $\mathbf{b}$ is denoted by $\mathbf{a} \otimes \mathbf{b}$. It is a matrix with entries $\{a_i b_j\}$. By $\mathbb{M}^{d\times d}$ we denote the space of real $d \times d$ matrices and $\mathbb{1}$ denotes the unit matrix (if $d = 2$ we use a special notation $\widehat{\mathbb{1}}$). Symmetric matrices form the subspace $\mathbb{M}_s^{d\times d}$. For $\mathsf{A} \in \mathbb{M}_s^{d\times d}$, the smallest and largest eigenvalues are denoted by $\lambda_{\ominus}(\mathsf{A})$ and $\lambda_{\oplus}(\mathsf{A})$, respectively. The *trace* and the *deviator* of $\boldsymbol{\sigma} \in \mathbb{M}^{d\times d}$ are defined by the formulas

$$\operatorname{tr}\boldsymbol{\sigma} := \sum_{i=1}^{d} \sigma_{ii} \quad \text{and} \quad \boldsymbol{\sigma}^D := \boldsymbol{\sigma} - \frac{1}{d}\operatorname{tr}\boldsymbol{\sigma}\,\mathbb{1}. \tag{1.1}$$

Since $\mathbb{1} : \boldsymbol{\sigma}^D = 0$, the above decomposition of $\boldsymbol{\sigma}$ is orthogonal and for any tensor $\boldsymbol{\sigma}$ we have the identity $|\boldsymbol{\sigma}|^2 = |\boldsymbol{\sigma}^D|^2 + \frac{1}{d}|\operatorname{tr}\boldsymbol{\sigma}|^2$.

By $[g]_\gamma$ we denote the jump (difference of the left-hand side and right-hand side limits of the function g) on a line (surface) γ.

1.1.2 Spaces of functions

Spaces of functions are denoted by capital letters X, Y, V. By default, all of them are assumed to be reflexive Banach spaces over the field of real numbers. The respective topologically dual spaces (which consist of linear continuous functionals) are marked by an asterisk (e.g., X^*, Y^*, V^*) and the duality pairings are denoted by round or angle brackets (e.g., (y^*, y) or $\langle v^*, v\rangle$). If V is a Banach space, then V^* can also be normed by setting

$$\|v^*\|_* := \sup_{v\in V\setminus\{0\}} \frac{\langle v^*, v\rangle}{\|v\|}.$$

For $\alpha \in [1, \infty]$, we denote by $L^\alpha(\Omega)$ the usual Lebesgue space of functions with norm $\|\cdot\|_{\alpha,\Omega}$. If $\alpha = 2$ then we may also use the simplified notation $\|\cdot\|_\Omega$ and for the scalar product $(\cdot,\cdot)_\Omega$. $L^2(\Omega, \mathbb{R}^d)$ is the Hilbert space of vector-valued functions, whose components are square integrable in Ω. The analogous space of tensor-valued functions is denoted $L^2(\Omega, \mathbb{M}^{d\times d})$.

By $\{\!|g|\!\}_\omega$ we denote the mean value of $g \in L^1(\omega)$ in ω, i.e.,

$$\{\!|g|\!\}_\omega := \frac{1}{|\omega|}\int_\omega g\,d\mathbf{x}, \tag{1.2}$$

where $|\omega|$ is the Lebesgue measure of the set ω. For vectors and matrices, the symbol $\{\!\!\{\cdot\}\!\!\}$ means componentwise averaging.

$\widetilde{L}^2(\Omega)$ denotes a subspace of $L^2(\Omega)$ consisting of the functions with zero mean values and $L^\infty(\Omega)$ is the space of functions bounded almost everywhere in Ω, endowed with the supremum norm $\|\cdot\|_{\infty,\Omega}$. If $\mathbf{f}$ is a vector-valued function, than

$$\|\mathbf{f}\|_{\infty,\omega} := \operatorname*{ess\,sup}_{\mathbf{x}\in\omega} |\mathbf{f}(\mathbf{x})|\,,$$

where $|\mathbf{f}|$ is the Euclidean norm. Whenever a different norm will be used this fact will be specially mentioned.

For $1 \leq p \leq \infty$, $|\cdot|_{\ell^p}$ denotes the discrete ℓ^p-norm in $\mathbb{R}^d$. If $p = 2$, then we use $|\cdot|$ instead of $|\cdot|_{\ell^2}$. For any $p \in [1,\infty]$, the conjugate number p' is defined by the relation $\frac{1}{p} + \frac{1}{p'} = 1$ (in some formulas the adjoint numbers are marked by stars, e.g., $p^\star$). Analogously, for $p \in [2,\infty]$, the number $p'' \in [1,\infty]$ satisfies the relation $\frac{2}{p} + \frac{1}{p''} = 1$.

$P^k(\Omega)$ is the space of polynomials of maximal degree k defined in Ω.

For derivatives, we use the standard notation (e.g., $\frac{\partial f}{\partial x_1}$ and $\frac{\partial^2 G}{\partial x_1 \partial x_2}$). In some parts, we also apply a shortened notation, where the directions of differentiation are shown in subscripts (e.g., $f_{,1}$ and $G_{,12}$). Also, in expressions containing multiindexes, we use Einstein's convention on summation over the repeated indices, e.g., $u_i v_i$ (where $i \in \{1,2,\ldots,d\}$) means the sum $\sum_{i=1}^d u_i v_i$.

$C^k(\Omega)$ denotes the space of k-times differentiable scalar-valued functions and $C_0^k(\Omega)$ is the subspace consisting of the functions with compact support in Ω. $C_0^\infty(\Omega)$ is the space of all infinitely differentiable functions with compact support in Ω.

In the book, we use standard differential operators: gradient (∇), curl, and div. The divergence of a tensor-valued function $\boldsymbol{\tau}$ is denoted by Div $\boldsymbol{\tau}$. It is defined by the vector $(\operatorname{div}\tau_i)_{i=1}^d$, where τ_j is the j-th row of $\boldsymbol{\tau}$.

$\mathrm{S}(\Omega)$ denotes the set of solenoidal (divergence-free) vector-valued functions defined in Ω and $\overset{\circ}{\mathrm{S}}(\Omega,\mathbb{R}^d)$ denotes the closure of the set of smooth divergence-free functions vanishing on the boundary with respect to the norm of $H^1(\Omega,\mathbb{R}^d)$.

Standard Sobolev spaces of functions having in Ω generalized derivatives up to the order l in $L^p(\Omega)$ are denoted $W^{l,p}(\Omega)$ and $\|\cdot\|_{l,p,\Omega}$ denotes the respective norm. Similar notation is used for spaces of vector- and tensor-valued functions. Also, for a sufficiently smooth vector field $\mathbf{v} : \Omega \to \mathbb{R}^d$ we define the norms

$$\|\nabla\mathbf{v}\|^2_{1,2,\Omega} := \|\nabla\mathbf{v}\|^2_\Omega + \sum_{l,k=1}^d \left\|\mathbf{v}_{,lk}\right\|^2_\Omega,$$

$$\|\mathbf{v}\|^2_{2,2,\Omega} := \sum_{i=1}^d \|v_i\|^2_{2,2,\Omega} = \|\mathbf{v}\|^2_\Omega + \|\nabla\mathbf{v}\|^2_{1,2,\Omega}.$$

If $p = 2$, then for Sobolev spaces we use the simplified notation $H^l(\Omega)$. The subspace of $H^1(\Omega)$ consisting of the functions that vanish on the boundary is denoted

$\overset{\circ}{H}{}^1(\Omega)$. $H^{-1}(\Omega)$ is the space dual to $\overset{\circ}{H}{}^1(\Omega)$. The space $W^{-1,p}(\Omega)$ is dual to $\overset{\circ}{W}{}^{1,p'}(\Omega)$. It is endowed with the standard dual norm $\| \cdot \|_{-1,p,\Omega}$.

$H(\mathrm{div}, \Omega)$ denotes the Hilbert space of square-integrable vector-valued functions with square-integrable divergence, endowed with the scalar product and the norm

$$(\mathbf{u}, \mathbf{v})_{\mathrm{div}} := \int_\Omega (\mathbf{u} \cdot \mathbf{v} + \mathrm{div}\, \mathbf{u}\, \mathrm{div}\, \mathbf{v}) d\mathbf{x}, \quad \|\mathbf{v}\|_{\mathrm{div}} := (\mathbf{v}, \mathbf{v})_{\mathrm{div}}^{1/2}.$$

Analogously, $H(\mathrm{Div}, \Omega)$ denotes the Hilbert space of square-integrable tensor-valued functions with square-integrable divergence and

$$(\boldsymbol{\tau}, \boldsymbol{\sigma})_{\mathrm{Div}} := \int_\Omega (\boldsymbol{\tau} : \boldsymbol{\sigma} + \mathrm{div}\, \boldsymbol{\tau} \cdot \mathrm{div}\, \boldsymbol{\sigma}) d\mathbf{x}, \quad \|\boldsymbol{\sigma}\|_{\mathrm{Div}} = (\boldsymbol{\sigma}, \boldsymbol{\sigma})_{\mathrm{Div}}^{1/2}.$$

Let $\mathsf{M} \in L^\infty\left(\Omega, \mathbb{M}_s^{d\times d}\right)$. We define

$$\rho(\mathsf{M}) := \operatorname*{ess\,sup}_{\mathbf{x}\in\Omega} \left(\sup_{\zeta \in \mathbb{R}^d \setminus \{0\}} \frac{\|\mathsf{M}(\mathbf{x})\zeta\|}{\|\zeta\|} \right). \tag{1.3}$$

For $p \geq 2$, we introduce the function $m \in L^\infty(\Omega)$ by

$$m := \sup_{\zeta \in \mathbb{R}^d \setminus \{0\}} \frac{\|\mathsf{M}(\cdot)\, \zeta\|_{\ell^{p'}}}{\|\zeta\|_{\ell^p}} \quad \text{and the norm} \quad ||| \, \mathsf{M} \, |||_{p'',\Omega} = \|m\|_{p'',\Omega}.$$

If $p = 2$ then $p' = 2$, $p'' = \infty$, and

$$||| \, M \, |||_{\infty,\Omega} = \rho(M). \tag{1.4}$$

We say that a matrix function $\mathsf{B} \in L^\infty\left(\Omega, \mathbb{M}_s^{d\times d}\right)$ is *uniformly positive definite* if $\mathsf{B}(x)$ is positive definite for all $x \in \Omega$ and

$$0 < \lambda_\ominus^\infty(\mathsf{B}) := ||| \, \mathsf{B}^{-1} \, |||_{\infty,\Omega}^{-1} \leq ||| \, \mathsf{B} \, |||_{\infty,\Omega} =: \lambda_\oplus^\infty(\mathsf{B}) < \infty \tag{1.5}$$

and define the *spectral condition number*

$$\kappa_{\mathsf{B}} := \lambda_\oplus^\infty(\mathsf{B}) / \lambda_\ominus^\infty(\mathsf{B}). \tag{1.6}$$

1.1.3 Convex functionals

A set $K \subset V$ is called *convex* if $\lambda v_1 + (1-\lambda) v_2 \in K$ for any $v_1, v_2 \in K$ and $\lambda \in [0, 1]$. $\mathrm{Conv}(K)$ denotes a smallest convex set containing K. It is called the *convex hull* of K.

Let K be a convex set in a Banach space V. A functional $\mathcal{I} : K \to \overline{\mathbb{R}}$ is called *convex* if

$$\mathcal{I}(\lambda_1 v_1 + \lambda_2 v_2) \leq \lambda_1 \mathcal{I}(v_1) + \lambda_2 \mathcal{I}(v_2) \tag{1.7}$$

for all $v_1, v_2 \in K$ and all $\lambda_1, \lambda_2 \in \mathbb{R}_{\geq 0}$ such that $\lambda_1 + \lambda_2 = 1$. It is called *strictly convex* if for positive λ_i, $i = 1, 2$ the inequality is strict. A functional $\mathcal{I}$ is called *concave* (resp., *strictly concave*) if the functional $-\mathcal{I}$ is convex (resp., strictly convex).

The *characteristic functional* of the the set K

$$\chi_K(v) = \begin{cases} 0, & \text{if } v \in K, \\ +\infty, & \text{if } v \notin K, \end{cases} \tag{1.8}$$

is convex if and only if K is a convex set.

The functional $\mathcal{I}^* : V^* \to \overline{\mathbb{R}}$ defined by the relation

$$\mathcal{I}^*(v^*) = \sup_{v \in V} \{\langle v^*, v\rangle - \mathcal{I}(v)\} \tag{1.9}$$

is called *dual* (or *conjugate*) to $\mathcal{I}$ (see, e.g., [103, 110, 280]). For example, the functional χ^*_K conjugate to χ_K is a cone in the space V^*, called the *support functional* of the set K.

If $V = \mathbb{R}$ and $\mathcal{I}$ is a smooth function, then $\mathcal{I}^*$ coincides with the Legendre transform of $\mathcal{I}$.

The second conjugate is defined by the relation

$$\mathcal{I}^{**}(v) := \sup_{v^* \in V^*} \{\langle v^*, v\rangle - \mathcal{I}^*(v^*)\}.$$

If V is a reflexive Banach space and $\mathcal{I}$ is convex, then $\mathcal{I}^{**}$ coincides with $\mathcal{I}$.

By definition,

$$\langle v^*, v\rangle \leq \mathcal{I}(v) + \mathcal{I}^*(v^*). \tag{1.10}$$

For example, if $V = \mathbb{R}^d$ and $\mathcal{I}(\mathbf{v}) = \frac{1}{\alpha}|\mathbf{v}|^\alpha$, then $\mathcal{I}^*(\mathbf{v}^*) = \frac{1}{\alpha^*}|\mathbf{v}^*|^{\alpha^*}$, where α and α^* are positive real numbers such that $\frac{1}{\alpha^*} + \frac{1}{\alpha} = 1$ (these numbers are called conjugate). In this case (1.10) reads

$$\mathbf{v} \cdot \mathbf{v}^* \leq \frac{1}{\alpha}|\mathbf{v}|^\alpha + \frac{1}{\alpha^*}|\mathbf{v}^*|^{\alpha^*}. \tag{1.11}$$

This inequality is also known as the Young inequality. It also holds for the space $V = \mathbb{M}^{d\times d}$ endowed with the Frobenius matrix norm.

From (1.11) we deduce the inequality

$$|\mathbf{v}_1 + \mathbf{v}_2|^2 \leq (1+\beta)|\mathbf{v}_1|^2 + \frac{1+\beta}{\beta}|\mathbf{v}_2|^2, \tag{1.12}$$

valid for any $\beta > 0$ and any pair of vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ in $\mathbb{R}^d$. Setting $\gamma_1 = 1 + \beta$ and $\gamma_2 = \frac{1+\beta}{\beta}$, we rewrite this inequality in the somewhat different form

$$|\mathbf{v}_1 + \mathbf{v}_2|^2 \leq \gamma_1|\mathbf{v}_1|^2 + \gamma_2|\mathbf{v}_2|^2. \tag{1.13}$$

Clearly (1.11), (1.12), and (1.12) can be extended to spaces of functions. Let V be a Hilbert space with the norm $\|\cdot\|_V$. For any v_1, $v_2 \in V$, we have

$$\|v_1 + v_2\|^2_V \leq \gamma_1\|v_1\|^2_V + \gamma_2\|v_2\|^2_V, \tag{1.14}$$

where γ_1 and γ_2 are positive numbers such that $\frac{1}{\gamma_1} + \frac{1}{\gamma_2} = 1$.

For a given $v_0 \in V$, an element $v^* \in V^*$ satisfying

$$\langle v^*, v - v_0\rangle + \mathcal{I}(v_0) \le \mathcal{I}(v) \qquad \forall\, v \in V \tag{1.15}$$

is called a *subgradient* of $\mathcal{I}$ at v_0. The set of all subgradients of $\mathcal{I}$ at v_0 forms the *subdifferential* $\partial\mathcal{I}(v_0)$. By $\mathcal{I}'(v_0)$ we denote an element $v^* \in V^*$ such that the derivative of $\mathcal{I}$ at $v_0 \in V$ in the direction w has the form $\langle v^*, w\rangle$ for any $w \in V$. This element is called the Gâteaux derivative of $\mathcal{I}$ at v_0 (if this notation is used, then it is assumed that the derivative exists).

The functional $\mathcal{D}_{\mathcal{I}} : V \times V^* \to \mathbb{R}$ defined by the relation

$$\mathcal{D}_{\mathcal{I}}(v, v^*) := \mathcal{I}(v) + \mathcal{I}^*(v^*) - \langle v^*, v\rangle,$$

where $\mathcal{I}$ and $\mathcal{I}^*$ are conjugate functionals, is called a *compound* functional. These functionals play an important role in error analysis of nonlinear problems. Throughout the book, we denote them by the letter $\mathcal{D}$ supplied with an index that shows the functional used to form it. In view of (1.10), $\mathcal{D}_{\mathcal{I}}$ is nonnegative. Moreover, it vanishes only if v and v^* satisfy the subdifferential (duality) relations (see, e.g., [103])

$$v \in \partial\mathcal{I}^*(v^*) \text{ and } v^* \in \partial\mathcal{I}(v). \tag{1.16}$$

In general, compound functionals are not convex. However, they possess a certain property similar to convexity. Let λ_1 and λ_1^* be real numbers in [0, 1] and $\lambda_2 = 1-\lambda_1$, $\lambda_2^* = 1 - \lambda_1^*$. For any $y_1, y_2 \in Y$ and $y_1^*, y_2^* \in Y^*$, we have

$$\begin{aligned} \mathcal{D}_{\mathcal{I}}\big(\lambda_1 y_1 + \lambda_2 y_2, \lambda_1^* y_1^* + \lambda_2^* y_2^*\big) \le \lambda_1\lambda_1^*\mathcal{D}_{\mathcal{I}}(y_1, y_1^*) + \lambda_1\lambda_2^*\mathcal{D}_{\mathcal{I}}(y_1, y_2^*) \\ + \lambda_2\lambda_1^*\mathcal{D}_{\mathcal{I}}(y_2, y_1^*) + \lambda_2\lambda_2^*\mathcal{D}_{\mathcal{I}}(y_2, y_2^*). \end{aligned} \tag{1.17}$$

Indeed,

$$\begin{aligned} \mathcal{D}_{\mathcal{I}}\left(y, \lambda_1^* y_1^* + \lambda_2^* y_2^*\right) &= \mathcal{I}(y) + \mathcal{I}^*(\lambda_1^* y_1^* + \lambda_2^* y_2^*) - (\lambda_1^* y_1^* + \lambda_2^* y_2^*, y) \\ &\le \mathcal{I}(y) + \lambda_1^*\mathcal{I}^*(y_1^*) + \lambda_2^*\mathcal{I}^*(y_2^*) - (\lambda_1^* y_1^* + \lambda_2^* y_2^*, y) \\ &= \lambda_1^*\mathcal{D}_{\mathcal{I}}(y, y_1^*) + \lambda_2^*\mathcal{D}_{\mathcal{I}}(y, y_2^*). \end{aligned} \tag{1.18}$$

Analogously

$$\mathcal{D}_{\mathcal{I}}\big(\lambda_1 y_1 + \lambda_2 y_2, y^*\big) \le \lambda_1\mathcal{D}_{\mathcal{I}}(y_1, y^*) + \lambda_2\mathcal{D}_{\mathcal{I}}(y_2, y^*). \tag{1.19}$$

Therefore,

$$\begin{aligned} &\mathcal{D}_{\mathcal{I}}(\lambda_1 y_1 + \lambda_2 y_2, \lambda_1^* y_1^* + \lambda_2^* y_2^*) \\ &\qquad \le \lambda_1\mathcal{D}_{\mathcal{I}}(y_1, \lambda_1^* y_1^* + \lambda_2^* y_2^*) + \lambda_2\mathcal{D}_{\mathcal{I}}(y_2, \lambda_1^* y_1^* + \lambda_2^* y_2^*)) \end{aligned}$$

and (1.17) follows from (1.18) and (1.19).

Remark 1.1.1. *From (1.19) it follows that for any $z_1, z_2 \in Y$ and $y^* \in Y^*$*

$$\mathcal{D}_{\mathcal{I}}\big(z_1 + z_2, y^*\big) \le \lambda_1\mathcal{D}_{\mathcal{I}}\left(\frac{z_1}{\lambda_1}, y^*\right) + \lambda_2\mathcal{D}_{\mathcal{I}}\left(\frac{z_2}{\lambda_2}, y^*\right). \tag{1.20}$$

Similarly, for any $z_1^*, z_2^* \in Y^*$ and $y \in Y$

$$\mathcal{D}_{\mathcal{I}}\big(y, z_1^* + z_2^*\big) \le \lambda_1^*\mathcal{D}_{\mathcal{I}}\left(y, \frac{z_1^*}{\lambda_1^*}\right) + \lambda_2^*\mathcal{D}_{\mathcal{I}}\left(y, \frac{z_2^*}{\lambda_2^*}\right). \tag{1.21}$$

1.2 Functional inequalities

For functions in Sobolev spaces, there exists a wide collection of so-called embedding inequalities (see, e.g., S. L. Sobolev [295], O. A. Ladyzhenskaya and N. N. Uraltseva [168], D. Gilbarg and N. S. Trudinger [120], R. A. Adams and J. J. Fournier [4]). They are of crucial importance for both qualitative and quantitative analysis of partial differential equations. For the convenience of the reader we discuss briefly below some of the results used in subsequent chapters. A systematic overview of sharp estimates of constants in various functional inequalities is presented in [162].

1.2.1 Hölder type inequalities

The discrete Hölder inequality

$$|\mathbf{a}\cdot\mathbf{b}| \leq \left(\sum_{i=1}^{d} |a_i|^{\alpha}\right)^{1/\alpha} \left(\sum_{i=1}^{d} |b_i|^{\alpha^*}\right)^{1/\alpha^*} \tag{1.22}$$

holds for $\mathbf{a}, \mathbf{b} \in \mathbb{R}^d$. For $w \in L^{\alpha}(\omega)$ and $v \in L^{\alpha^*}(\omega)$, $\alpha \in [1, +\infty]$, where ω is a bounded Lipschitz domain, the integral Hölder inequality reads

$$\int_{\omega} w\, v\, d\mathbf{x} \leq \|w\|_{\alpha,\omega} \|v\|_{\alpha^*,\omega}. \tag{1.23}$$

Similar inequalities hold for vector- and matrix-valued functions. For instance, if $\boldsymbol{\sigma} \in L^{\alpha}(\Omega, \mathbb{M}^{d\times d})$ and $\boldsymbol{\tau} \in L^{\alpha^*}(\Omega, \mathbb{M}^{d\times d})$, then

$$\int_{\omega} \boldsymbol{\sigma} : \boldsymbol{\tau}\, d\mathbf{x} \leq \|\boldsymbol{\sigma}\|_{\alpha,\omega} \|\boldsymbol{\tau}\|_{\alpha^*,\omega}. \tag{1.24}$$

We will also use the following multiplicative estimate, which is valid for scalar and vector valued functions.

Let $2 < r < t < +\infty$ and $\theta(r,t) := \frac{2(t-r)}{r(t-2)} \in (0,1)$. For $w \in L^t(\Omega)$,

$$\|w\|_{r,\Omega} \leq \|w\|_{2,\Omega}^{\theta(r,t)} \|w\|_{t,\Omega}^{1-\theta(r,t)}. \tag{1.25}$$

A similar inequality holds for vector-valued functions. Hence for $w \in W^{1,t}(\Omega)$, we have

$$\|\nabla w\|_{r,\Omega} \leq \|\nabla w\|_{2,\Omega}^{\theta(r,t)} \|\nabla w\|_{t,\Omega}^{1-\theta(r,t)}. \tag{1.26}$$

1.2.2 Friedrichs and Poincaré inequalities

Let $\ell : W^{1,p}(\Omega) \to \mathbb{R}$ ($p \in [1, +\infty)$ be a linear continuous functional satisfying the condition: if $\ell(w) = 0$ for any constant function w, then $w = 0$. In this case, the original norm of $W^{1,p}(\Omega)$ is equivalent to the norm $|\ell(w)| + \|\nabla w\|_{p,\Omega}$ (this fact is proved with the help of the compactness method). Since $W^{1,p}(\Omega)$ is embedded in $L^p(\Omega)$, we conclude that

$$\|w\|_{p,\Omega} \le C(p,\Omega,d)\left(|\ell(w)| + \|\nabla w\|_{p,\Omega}\right) \qquad \forall\, w \in W^{1,p}(\Omega). \tag{1.27}$$

Particular forms of (1.27) arise if w belongs to the subspace of $W^{1,p}(\Omega)$ defined by the condition $\ell(w) = 0$. Then, (1.27) reads

$$\|w\|_{p,\Omega} \le C(p,\Omega,d)\|\nabla w\|_{p,\Omega} \quad \forall\, w \in \{W^{1,p}(\Omega) \mid w \in \ker \ell\}. \tag{1.28}$$

In our analysis, we need guaranteed and explicitly computable estimates of the constant $C(p,\Omega,d)$. Henceforth, for simplicity we often use a shorter notation $C(\Omega)$ for such type constants.

The Poincaré inequality If $\ell(w) = \int\limits_\Omega w\, d\mathbf{x}$, then the set $\ker \ell$ consists of the functions satisfying the condition $\{\!\!\{w\}\!\!\}_\Omega = 0$ and (1.28) yields

$$\|w\|_{p,\Omega} \le C_{\mathrm{P}}(\Omega)\|\nabla w\|_{p,\Omega} \quad \forall\, w \in \widetilde{W}^{1,p}(\Omega), \tag{1.29}$$

where

$$\widetilde{W}^{1,p}(\Omega) := \{W^{1,p}(\Omega) \mid \{\!\!\{w\}\!\!\}_\Omega = 0\}.$$

If $p = 2$, we obtain the classical inequality established by H. Poincaré [239] (originally for convex domains with smooth boundaries). For piecewise smooth domains this inequality (and a similar inequality for functions vanishing on the boundary) was independently established by V. Steklov [298], who proved that $C_{\mathrm{P}} = \lambda^{-\frac{1}{2}}$, where λ is the smallest positive eigenvalue of the problem

$$-\Delta u = \lambda u \quad \text{in} \quad \Omega, \tag{1.30}$$

$$\frac{\partial u}{\partial \mathbf{n}} = 0 \quad \text{on} \quad \partial\Omega. \tag{1.31}$$

Getting guaranteed and computable bounds of C_{P} (and other constants in various functional inequalities; see, e.g., S. Mikhlin [203]) is a question of utmost importance for quantitative analysis of partial differential equations. Sometimes this question can be answered fairly easily. The very first estimates of C_{P} was actually obtained by H. Poincaré ($C_{\mathrm{P}}(\Omega) \le 0.5401\,\mathrm{diam}\,\Omega$ for $d = 2$, where $\mathrm{diam}\,\Omega$ denotes the diameter of Ω). In general, finding the constant is equivalent to finding a lower bound of the smallest positive eigenvalue associated with some differential problem (as in (1.10)–(1.11)). Such a problem may be rather difficult. Below we briefly discuss different results that help to overcome these difficulties.

If Ω is a convex domain and $p = 2$, then for any d we have the following easily computable upper bound of the constant (see L. Payne and H. Weinberger [232]):

$$C_{\mathrm{P}}(\Omega) \le \frac{\mathrm{diam}\,\Omega}{\pi} \approx 0.3183\ \mathrm{diam}\,\Omega. \tag{1.32}$$

A lower bound of $C_P(\Omega)$ was derived in S. Cheng [82] (for $d = 2$):

$$C_P(\Omega) \geq \frac{\operatorname{diam}\Omega}{2 j_{0,1}} \approx 0.2079 \operatorname{diam}\Omega. \tag{1.33}$$

Here $j_{0,1} \approx 2.4048$ is the smallest positive root of the Bessel function J_0.

For *isosceles triangles* an improvement of the upper bound is due to R. S. Laugesen and B. A. Siudeja [174], who proved that

$$C_P(\Omega) \leq \operatorname{diam}\Omega \begin{cases} \frac{1}{j_{1,1}}, & \text{if} \quad \alpha \leq \frac{\pi}{3}, \\ \min\{\frac{1}{j_{1,1}}, \frac{1}{j_{0,1}}(2(\pi-\alpha)\tan(\alpha/2))^{-1/2}\}, & \text{if} \quad \alpha \in (\frac{\pi}{3}, \frac{\pi}{2}], \\ \frac{1}{j_{0,1}}(2(\pi-\alpha)\tan(\alpha/2))^{-1/2}, & \text{if} \quad \alpha \in (\frac{\pi}{2}, \pi), \end{cases} \tag{1.34}$$

and $j_{1,1} \approx 3.8317$ is the smallest positive root of the Bessel function J_1.

G. Acosta and R. Duran [3], have shown that for convex domains the constant in the L_1 Poincaré type inequality satisfies the estimate

$$\inf_{c\in\mathbb{R}} \|w - c\|_{1,\Omega} \leq \frac{\operatorname{diam}\Omega}{2} \|\nabla w\|_{1,\Omega}. \tag{1.35}$$

Estimates of the constant for other p can be found in S.-K. Chua and R. L. Wheeden [87] (also for convex domains).

The Friedrichs inequality Another important case is when the functional ℓ is defined by the trace operator, so that the condition $\ell(w) = 0$ defines a subspace $\overset{\circ}{H}{}^1(\Omega)$ containing functions vanishing on $\partial\Omega$ (or a part of $\partial\Omega$ with positive boundary measure). Then we arrive at the Friedrichs inequality

$$\|w\|_\Omega \leq C_F(\Omega)\|\nabla w\|_\Omega \quad \forall w \in \overset{\circ}{H}{}^1(\Omega). \tag{1.36}$$

Analogous estimates hold for L^p norms $p \in [1, +\infty)$ (see, e.g., [120]) provided that w is a function in $W^{1,p}(\Omega)$ vanishing on the boundary.

It is easy to show that the constant in (1.36) is defined by the lowest eigenvalue of the operator Δ, which satisfies the Rayleigh relation

$$\frac{1}{C_F^2(\Omega)} = \lambda_\Omega := \inf_{\substack{w\in\overset{\circ}{H}{}^1(\Omega)\\ w\neq 0}} \frac{\|\nabla w\|^2}{\|w\|^2}. \tag{1.37}$$

Therefore, lower estimates of the minimal eigenvalue generate upper estimates of the Friedrichs constant, and vice versa.

An upper bound of $C_F(\Omega)$ is easy to find by means of monotonicity arguments if the homogeneous boundary condition is imposed on the whole boundary $\partial\Omega$. Let

$\Omega \subset \Omega_+$. For any $w \in \overset{\circ}{H}{}^1(\Omega)$, we can define $\widehat{w} \in \overset{\circ}{H}{}^1(\Omega_+)$ by setting $\widehat{w} = w$ in Ω and $\widehat{w}(\mathbf{x}) = 0$ for any $\mathbf{x} \in \Omega_+ \setminus \Omega$. Since

$$\|w\|_{\Omega_+} \leq C_{\mathrm{F}}(\Omega_+)\|\nabla w\|_{\Omega_+} \quad \forall w \in \overset{\circ}{H}{}^1(\Omega_+),$$

we see that $C_{\mathrm{F}}(\Omega) \leq C_{\mathrm{F}}(\Omega_+)$. This simple observation opens a way of deriving simple upper bounds for the Friedrichs constant by using known constants for some special domains. For example, if

$$\Omega \subset \Omega_+ := \{\mathbf{x} \in \mathbb{R}^d \mid a_i < x_i < b_i, \quad b_i - a_i = l_i,\, i = 1, \ldots, d\},$$

then

$$C_{\mathrm{F}}(\Omega) \leq C_{\mathrm{F}}(\Omega_+) = \frac{1}{\pi}\left(\sum_{i=1}^{d} \frac{1}{l_i^2}\right)^{-1}. \tag{1.38}$$

For problems with mixed boundary conditions, the monotonicity approach is not applicable. However, there exist numerical methods that generate lower bounds of eigenvalues (see [54, 79, 290, 312] and references therein) and upper bounds of the respective constants. Also, we note that discrete versions of the Friedrichs and Poincaré inequalities valid for piecewise H^1 functions are established in [67]. They are often used in error analysis of various nonconforming approximations (e.g., see [216]).

1.2.3 Inequalities for functions with zero mean traces on the boundary

In some cases, the following advanced forms of the Poincaré estimate are useful. Let Γ be a measurable part of $\partial\Omega$ (we assume that the surface measure of Γ is positive) and

$$\widetilde{H}^1(\Omega, \Gamma) := \left\{ w \in V := H^1(\Omega) \mid \{\!|w|\!\}_\Gamma = \frac{1}{|\Gamma|}\int_\Gamma w\, ds = 0 \right\},$$

It is clear that the linear functional $\ell_\Gamma(w) := \int_\Gamma w\, ds$ satisfies the condition

$$\ell_\Gamma(w) = 0 \Rightarrow w = 0 \text{ for any } w \in P^0.$$

Therefore, we have the estimates

$$\|w\|_{2,\Omega} \leq C_1(\Omega, \Gamma)\|\nabla w\|_{2,\Omega}, \qquad \forall w \in \widetilde{H}^1(\Omega, \Gamma), \tag{1.39}$$

$$\|w\|_{2,\Gamma} \leq C_2(\Omega, \Gamma)\|\nabla w\|_{2,\Omega}, \qquad \forall w \in \widetilde{H}^1(\Omega, \Gamma). \tag{1.40}$$

Exact constants $C_1(\Omega, \Gamma)$ and $C_2(\Omega, \Gamma)$ are known for some basic domains (rectangles, parallelepipeds, right triangles; see [209]). For example, if Ω is a rectangle $\Pi_{h_1 \times h_2} := (0, h_1) \times (0, h_2)$ and $\Gamma = \{x_1 = 0, x_2 \in [0, h_2]\}$, then

$$C_1 = \frac{\max\{2h_1, h_2\}}{\pi} \quad \text{and} \quad C_2 = \left(\frac{\pi}{h_2}\tanh\left(\frac{\pi h_1}{h_2}\right)\right)^{-1/2}. \tag{1.41}$$

If Ω is a parallelepiped $\Pi_{h_1\times h_2\times h_3} := (0,h_1)\times(0,h_2)\times(0,h_3)$ and Γ is the face defined by the condition $x_1 = 0$, then

$$C_1 = \frac{\max\{2h_1, h_2, h_3\}}{\pi} \quad \text{and} \quad C_2 = (\zeta \tanh(\zeta h_1))^{-1/2}, \tag{1.42}$$

where $\zeta = \frac{\pi}{\max\{h_2;h_3\}}$.

If $\Omega = \{0 < x_2 < x_1 < h\}$ and $\Gamma = \{x_1 = h,\ x_2 \in [0,h]\}$ (i.e., Γ is a cathetus of the right triangle), then $C_1 = h\zeta^{-1}$, where $\zeta \approx 2.02876$ is the unique root of the equation $\zeta\cot\zeta + 1 = 0$ in $(0,\pi)$ and $C_2 = \left(\frac{\zeta}{h}\tanh\zeta\right)^{-1/2}$, where $\zeta \approx 2.3650$ is the unique root of the equation $\tan\zeta + \tanh\zeta = 0$ in $(0,\pi)$.

A wider class of domains is considered in [190], where estimates of $C_{\rm P}$, $C_1(\Omega,\Gamma)$, and $C_2(\Omega,\Gamma)$ are deduced for convex polygonal and polyhedral domains. Applications of these type estimates to a posteriori error estimation for elliptic and parabolic problems are discussed on [259, 189, 191].

1.2.4 Korn's inequalities

Korn's inequalities [157] (first and second) establish the coercivity of bilinear forms generated by the linearised deformation tensor in continuum mechanics. For a bounded Lipschitz domain Ω, the second Korn inequality states that

$$\int_\Omega \left(|\mathbf{w}|^2 + |\boldsymbol{\varepsilon}(\mathbf{w})|^2\right) d\mathbf{x} \geq C_{\rm K}(\Omega)\|\mathbf{w}\|^2_{1,2,\Omega} \quad \forall \mathbf{w} \in H^1(\Omega,\mathbb{R}^d), \tag{1.43}$$

where $C_{\rm K}(\Omega)$ is a constant independent of $\mathbf{w}$ and $\boldsymbol{\varepsilon}(\mathbf{w}) := \frac{1}{2}\left(\nabla\mathbf{w} + (\nabla\mathbf{w})^T\right)$.

The kernel of $\boldsymbol{\varepsilon}(\mathbf{w})$ is the *space of rigid motions* $\mathbf{R}(\Omega)$. Any vector field $\mathbf{w} \in \mathbf{R}(\Omega)$ has the form $\mathbf{w} = \mathbf{w}_0 + \omega_0\mathbf{x}$, where $\mathbf{w}_0$ is a vector independent of $\mathbf{x} \in \mathbb{R}^d$, and ω_0 is a skew-symmetric tensor with coefficients independent of $\mathbf{x}$, $\dim \mathbf{R}(\Omega) = \frac{d(d+1)}{2}$.

In general, finding the constant $C_{\rm K}(\Omega)$ may be a very difficult problem. One exception is related to the case of homogeneous Dirichlet boundary conditions. For $\mathbf{w} \in \overset{\circ}{H}{}^1(\Omega)$, it is easy to show that

$$\|\nabla\mathbf{w}\| \leq \sqrt{2}\|\boldsymbol{\varepsilon}(\mathbf{w})\|. \tag{1.44}$$

The Korn inequalities are well studied. First, we mention the classical work of Friedrichs [113] and subsequent publications [142, 144, 159, 212, 222, 75, 210] (see also the monographs [89, 102]). Korn-type inequalities for piecewise H^1 vector fields (which are important for certain classes of numerical approximations) were established in [68] and some interesting generalizations of the Korn inequality have been recently presented in [213].

For analysis of models in continuum mechanics we often need certain analogues of the Friedrichs and Poincaré inequalities valid for vector valued functions and the operator $\boldsymbol{\varepsilon}$. They are

$$\|\mathbf{w}\| \leq C_{\mathrm{F},\varepsilon}\|\boldsymbol{\varepsilon}(\mathbf{w})\| \qquad \forall \mathbf{w} \in V_{\Gamma_0}(\Omega), \tag{1.45}$$

$$\inf_{\mathbf{z}\in\mathbf{R}} \|\mathbf{w}-\mathbf{z}\| \leq C_{\mathrm{P},\varepsilon}\|\boldsymbol{\varepsilon}(\mathbf{w})\| \qquad \forall \mathbf{w} \in H^1(\Omega,\mathbb{R}^d), \tag{1.46}$$

where V_{Γ_0} denotes a subspace of $H^1(\Omega,\mathbb{R}^d)$ consisting of the functions that vanish on the boundary of Ω or on some measurable part Γ_0 with positive surface measure. The value of $C_{\mathrm{F},\varepsilon}$ (or $C_{\mathrm{P},\varepsilon}$) readily follows from the respective Friedrichs (Poincaré) constant and C_{K}. However, this method is applicable only provided that C_{K} is known. In Chapter 4, related to dimension reduction models, we suggest a way to bypass this difficulty for 3D plate-type domains, where a simpler majorant of the constant is deduced by using separation of variables.

1.2.5 Inf–Sup condition

Well-posedness of mathematical problems in the theory of viscous incompressible fluids is based on the following result.

Lemma 1.2.1 ([22, 71, 167]). *Let Ω be a bounded domain with Lipschitz continuous boundary. There exists a constant $\kappa_\Omega > 0$ (which depends only on Ω) such that for any function $f \in L^2(\Omega)$ satisfying the condition $\{\!\!\{ f \}\!\!\}_\Omega = 0$ one can find a vector-valued function $\mathbf{w}_f \in \overset{\circ}{H}{}^1(\Omega,\mathbb{R}^d)$ such that*

$$\operatorname{div} \mathbf{w}_f = f \quad \text{in } \Omega \tag{1.47}$$

and

$$\|\nabla \mathbf{w}_f\| \leq \kappa_\Omega \|f\|. \tag{1.48}$$

This lemma is also called the “stability lemma for the Stokes problem” or “existence of a bounded inverse to the operator div”. Also, (1.48) can be viewed as a form of the Nečas inequality [211] (for Lipschitz domains a simple proof of this fact can be found in [64]).

Thanks to the paper by C. Horgan and L. Payne [144], it is known that for simply connected domains in $d = 2$ the constants κ_Ω and $C_{\mathrm{K}}(\Omega)$ in (1.43) are joined by the relation

$$2\kappa_\Omega = C_{\mathrm{K}}(\Omega) = 2(1 + L_\Omega), \tag{1.49}$$

where L_Ω is the constant in the Friedrichs type inequality $\|u\|^2 \leq L_\Omega\|v\|^2$, which holds for an analytic function $u + iv$ provided that $\{\!\!\{ u \}\!\!\}_\Omega = 0$ (see [112]).

Lemma 1.2.1 can be extended to L^q spaces for $1 < q < +\infty$ (see [56, 237, 238, 118]), namely, for $f \in L^q(\Omega)$ satisfying $\{\!\!\{ f \}\!\!\}_\Omega = 0$, there exists $\mathbf{w}_f \in \overset{\circ}{W}{}^{1,q}(\Omega,\mathbb{R}^d)$ such that

$$\operatorname{div} \mathbf{w}_f = f \qquad \text{and} \qquad \|\nabla \mathbf{w}_f\|_q \leq \kappa_{\Omega,q}\|f\|_q. \tag{1.50}$$

Another form of Lemma 1.2.1 is known in the literature as the Inf–Sup (or *Ladyzhenskaya–Babuška–Brezzi* (LBB)) condition: there exists a positive constant c_Ω such that

$$\inf_{\substack{q\in\widetilde{L}^2(\Omega)\\ q\neq 0}} \sup_{\substack{\mathbf{w}\in\overset{\circ}{H}{}^1(\Omega,\mathbb{R}^d)\\ \mathbf{w}\neq 0}} \frac{\int_\Omega q \operatorname{div}\mathbf{w}\, d\mathbf{x}}{\|q\|\,\|\nabla\mathbf{w}\|} \geq c_\Omega. \tag{1.51}$$

It is easy to show that (1.51) holds with $c_\Omega = (\kappa_\Omega)^{-1}$. Indeed, for arbitrary $q \in \widetilde{L}^2(\Omega)$, we can find $\mathbf{w}_q$ such that $\operatorname{div}\mathbf{w}_q = q$ and $\|\nabla\mathbf{w}_q\| \leq \kappa_\Omega\|q\|$, which implies the required result. The condition (1.51) and its discrete analogues are used for proving the stability and convergence of numerical methods in various problems related to the theory of viscous incompressible fluids (e.g., in [70, 71] this condition was proved and used to justify the convergence of the so-called *mixed* methods, in which a boundary-value problem is reduced to a saddle-point problem).

Estimates of κ_Ω Estimates of κ_Ω for various domains are important for the quantitative analysis of incompressible media problems. It is not difficult to see that the constant c_Ω in (1.51) is nonnegative and cannot exceed 1 (hence $\kappa_\Omega \geq 1$). Moreover, $c_\Omega > 0$ for any bounded Lipschitz domain. For domains with cusps, c_Ω may be equal to zero. For a ball in $\mathbb{R}^d$, $c_\Omega = \frac{1}{\sqrt{d}}$ and for the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} < 1$, where $a < b$, the constant satisfies the estimate $c_\Omega^2 \leq \frac{a^2}{a^2+b^2}$. Estimates for a number of other domains can be found in [100, 85, 152]. The latter publication is mainly devoted to numerical computation of c_Ω (what may be not an easy task even for simple domains). A variational principle obtained for κ_Ω in [262] can help in constructing numerical approximations of this constant.

Estimates of c_Ω are also known for Lipschitz domains in $\mathbb{R}^2$, which are star-shaped with respect to a ball with center $\mathbf{x}_0$. Let r be the ray from $\mathbf{x}_0$ crossing Γ at $\mathbf{x}$. For almost all $\mathbf{x} \in \Gamma$, there exists a unique tangent line, which forms a positive angle $\theta \leq \pi/2$ with the ray r. The quantity $\Theta_\Omega := \max_{\mathbf{x}\in\Gamma}\theta(\mathbf{x})$ generates the first guaranteed lower bound that can be computed by simple geometrical analysis (see [144]):

$$c_\Omega \geq \sin\frac{\Theta_\Omega}{2}. \tag{1.52}$$

However, this bound may be rather pessimistic (e.g., for a square $\Theta_\Omega = \frac{\pi}{4}$ and, therefore, the estimate shows that $c_\Omega \geq \sin\frac{\pi}{8} \approx 0.0069$ and $\kappa_\Omega \leq 146$).

In [98], a significant improvement of these estimates was obtained for domains in $\mathbb{R}^2$ which are contained in a disc of radius R and are star-shaped with respect to a concentric disc of radius ρ. Specifically, it was shown that

$$c_\Omega \geq \frac{\kappa}{\sqrt{2}}\left(1 + \sqrt{1-\kappa^2}\right)^{-1/2}, \tag{1.53}$$

where $\kappa = \frac{\rho}{R}$.

For $d = 3$, explicit bounds of κ_Ω are known only for domains with sufficiently regular boundaries. In [231], it was shown that for star shaped domains in $\mathbb{R}^3$ with C^1 boundary presented in the form $r = r_0(\phi, \psi)$, where (r, ϕ, ψ) are spherical coordinates. Estimates of the constants κ_Ω and c_Ω for exterior domains have been recently obtained in [230].

Distance to the set of divergence free fields The constant κ_Ω arises in estimates of the distance between a function $\mathbf{v} \in \overset{\circ}{H}{}^1(\Omega, \mathbb{R}^d)$ and the space $\overset{\circ}{\mathrm{S}}(\Omega, \mathbb{R}^d)$ consisting of divergence-free (solenoidal) vector functions vanishing on the boundary if the distance is measured in terms of the H^1 norm.

In view of Lemma 1.2.1, for $f = \operatorname{div} \mathbf{v}$ there exists $\mathbf{w}_f \in \overset{\circ}{H}{}^1(\Omega, \mathbb{R}^d)$ such that $\|\nabla \mathbf{w}_f\| \le \kappa_\Omega \|f\|$ and $\operatorname{div} \mathbf{w}_f = f$. Hence the function $\mathbf{w}_0 := \mathbf{v} - \mathbf{w}_f$ belongs to the space $\overset{\circ}{\mathrm{S}}(\Omega, \mathbb{R}^d)$ and $\|\mathbf{v} - \mathbf{w}_0\| \le \kappa_\Omega \|f\|$. Therefore,

$$\operatorname{dist}(\mathbf{v}, \overset{\circ}{\mathrm{S}}(\Omega, \mathbb{R}^d)) := \|\nabla(\mathbf{v} - \Pi_{\overset{\circ}{\mathrm{S}}} \mathbf{v})\|_\Omega \le \kappa_\Omega \|\operatorname{div} \mathbf{v}\|_\Omega, \tag{1.54}$$

where $\Pi_{\overset{\circ}{\mathrm{S}}} : \overset{\circ}{H}{}^1(\Omega, \mathbb{R}^d) \to \overset{\circ}{\mathrm{S}}(\Omega, \mathbb{R}^d)$ is the orthogonal projector. This estimate also follows from (1.51) (see [254]).

Estimates of this type are important in the evaluation of the accuracy of numerical solutions, which satisfy the divergence-free conditions only approximately or in comparing solutions of models accounting for the incompressibility condition in different (weaker) forms (see Chapter 6). For domains with complicated boundaries and holes, it may be very difficult to find sharp and guaranteed majorants of the constant κ_Ω (especially for 3D domains). Therefore, there arises the question of how to get practically applicable versions of (1.54) for domains of such a type. To answer it, we use ideas of domain decomposition. Below we briefly discuss the corresponding method referring for a more detailed exposition to [258, 260] and some other publications cited therein.

Assume that Ω is decomposed into N non-overlapping Lipschitz subdomains Ω_i, $i = 1, 2, \ldots, N$ and $f \in L^q(\Omega)$ $(0 < q < 1)$ satisfy the conditions

$$\{\!| f |\!\}_{\Omega_i} = 0, \quad i = 1, 2, \ldots, N. \tag{1.55}$$

Using (1.50), we obtain the following result:

Lemma 1.2.2. *If f satisfies (1.55), then there exists $\mathbf{v}_f \in \overset{\circ}{W}{}^{1,q}(\Omega, \mathbb{R}^d)$ such that*

$$\operatorname{div} \mathbf{v}_f = f \quad \text{and} \quad \|\nabla \mathbf{v}_f\|^q_{\Omega,q} \le \sum_{i=1}^{N} \kappa^q_{\Omega_i,q} \|f\|^q_{\Omega_i,q}, \tag{1.56}$$

where $\kappa_{\Omega_i,q}$ are positive constants associated with subdomains Ω_i.

To prove this estimate, note that by (1.50) there exists $\mathbf{v}_{f,i} \in \overset{\circ}{W}{}^{1,q}(\Omega_i, \mathbb{R}^d)$ such that

$$\operatorname{div} \mathbf{v}_{f,i} = f \text{ in } \Omega_i \quad \text{and} \quad \|\nabla \mathbf{v}_{f,i}\|_{\Omega_i,q} \leq \kappa_{\Omega_i,q}\|f\|_{\Omega_i,q}.$$

Define $\mathbf{v}_f(\mathbf{x}) = \mathbf{v}_{f,i}(\mathbf{x})$ if $\mathbf{x} \in \Omega_i$. Then, $\mathbf{v}_f \in \overset{\circ}{W}{}^{1,q}(\Omega, \mathbb{R}^d)$, $\operatorname{div} \mathbf{v}_f = f$, and

$$\|\nabla \mathbf{v}_f\|^q_{\Omega,q} = \sum_{i=1}^{n} \|\mathbf{v}_{f,i}\|^q_{\Omega_i,q} \leq \sum_{i=1}^{n} \kappa^q_{\Omega_i,q}\|f\|^q_{\Omega_i,q}.$$

Lemma 1.2.2 yields an estimate of the distance from $\mathbf{v} \in W^{1,q}(\Omega, \mathbb{R}^d)$ to the set of divergence-free fields provided that v satisfy additional conditions

$$\{\!\!\{\operatorname{div} \mathbf{v}\}\!\!\}_{\Omega_i} = 0 \qquad i = 1, 2, \ldots, N. \tag{1.57}$$

Since $\{\!\!\{\operatorname{div} \mathbf{v}\}\!\!\}_{\Omega} = 0$, the vector-valued function $\mathbf{v}$ satisfies $\int_\Gamma \mathbf{v} \cdot \mathbf{n}\, ds = 0$ and, therefore, the nonhomogeneous boundary condition on Γ admits a divergence-free extension. Thus, by shifting we can reduce this case to the above discussed case with homogeneous boundary conditions.

Notice that the integral conditions (1.57) do not lead to essential technical difficulties provided that N is not too large. Indeed, if $\mathbf{v}$ does not satisfy the conditions (1.57) exactly, then it is easy to correct it by changing values of $\mathbf{v} \cdot \mathbf{n}$ on $\Gamma_{ij} = \Omega_i \cap \Omega_j$ and $\Gamma_1 \cap \Omega_i$. The corresponding procedure changes N parameters in the representation of $\mathbf{v}$ such that all the boundary integrals vanish.

Lemma 1.2.3. *Let* $\mathbf{v} \in W^{1,q}(\Omega, \mathbb{R}^d)$ *satisfy* (1.57) *and* $\operatorname{div} \mathbf{v} \in L^{\mu_i}(\Omega_i, \mathbb{R}^d)$*, where* $\mu_i \geq q$*,* $i = 1, 2, \ldots, N$*. Then, there exists* $\mathbf{v}_0 \in W^{1,q}(\Omega, \mathbb{R}^d)$ *such that* $\operatorname{div} \mathbf{v}_0 = 0$*,* $\mathbf{v} = \mathbf{v}_0$ *on* Γ*, and*

$$\|\nabla(\mathbf{v} - \mathbf{v}_0)\|_{\Omega,q} \leq \left(\sum_{i=1}^{N} \kappa^q_{\Omega_i,q} |\Omega_i|^{1-\frac{q}{\mu_i}} \|\operatorname{div} \mathbf{v}\|^q_{\Omega_i,\mu_i} \right)^{1/q}. \tag{1.58}$$

Remark 1.2.4. *If* $\operatorname{div} v$ *is bounded almost everywhere (which is typical for piecewise polynomial approximations), then* (1.58) *yields the estimate*

$$\|\nabla(\mathbf{v} - \mathbf{v}_0)\|^q_{\Omega,q} \leq \sum_{i=1}^{N} \kappa^q_{\Omega_i,q} |\Omega_i| \left(\operatorname{ess\,sup}_{\Omega_i} |\operatorname{div} \mathbf{v}|\right)^q. \tag{1.59}$$

1.3 Computable bounds of constants in functional inequalities

The computation of exact (minimal) constants in Poincaré, Friedrichs, and other functional inequalities may be a very difficult problem, especially for multi-connected domains with complicated boundaries. However, for quantitative analysis it is usually enough to have guaranteed and realistic bounds of these constants. Here we discuss a method (suggested in [257, 261], see also [263]) capable of providing them. In general, the main idea of this method is similar to the one that was used for the derivation of a posteriori estimates of functional type: *use the integration by parts formulas generated by a pair of adjoint differential operators in order to transform certain unknown integral expressions into computable ones.*

As a result, estimates of constants in functional inequalities contain "free functions" (i.e., have the same principal structure as the estimates derived for measuring distances to the exact solution of a problem).

Any choice of such free functions (and of the supplementary parameters) provides a guaranteed upper bound, but, certainly, getting a good bound requires a rational selection (which can be done by the direct minimization of the majorant with respect to the set of free functions and parameters). Advanced forms of the method are based on ideas of domain decomposition.

1.3.1 Constant in the Friedrichs inequality

Let Ω be a bounded domain in $\mathbb{R}^d$ whose boundary has two measurable non-intersecting parts Γ_1 and Γ_2. Our first goal is to estimate integral quantities associated with a function

$$v \in V_0 := \{v \in W^{1,\alpha}(\Omega) \mid v = 0 \text{ on } \Gamma_1\}, \quad \alpha > 1.$$

For this purpose, we use a vector-valued function $\boldsymbol{\tau}$ in the set

$$Q_\phi := \left\{\boldsymbol{\tau} \in L^{\alpha^*}(\Omega, \mathbb{R}^d) \mid \operatorname{div} \boldsymbol{\tau} = \phi(\mathbf{x}) \in L^\infty(\Omega),\ \boldsymbol{\tau} \cdot \mathbf{n} = 0 \text{ on } \Gamma_2\right\},$$

where the condition $\boldsymbol{\tau} \cdot \mathbf{n} = 0$ on Γ_2 is understood in the sense that

$$\ell_{\boldsymbol{\tau}}(w) := \int_\Omega (\nabla w \cdot \boldsymbol{\tau} + w \operatorname{div} \boldsymbol{\tau}) d\mathbf{x} = 0 \qquad \forall w \in V_0. \tag{1.60}$$

Notice that the set Q_ϕ is not empty if the equation $\Delta u = \phi$ with the boundary conditions $u = g$ on Γ_1 and $\nabla u \cdot \mathbf{n} = 0$ on Γ_2 has a solution in W^{1,α^*} for some g.

In view of (1.60), we have for any $v \in V_0$

$$\left|\int_\Omega \phi v \, d\mathbf{x}\right| = \left|\int_\Omega v \operatorname{div} \boldsymbol{\tau} \, d\mathbf{x}\right| = \left|\int_\Omega \boldsymbol{\tau} \cdot \nabla v \, d\mathbf{x}\right| \le \|\boldsymbol{\tau}\|_{\alpha^*} \|\nabla v\|_\alpha,$$

which yields the estimate

$$\left|\int_\Omega \phi v \, d\mathbf{x}\right| \le C(\alpha^*, \phi, \Omega) \|\nabla v\|_\alpha, \qquad \text{where} \qquad C(\alpha^*, \phi, \Omega) = \inf_{\boldsymbol{\tau} \in Q_\phi} \|\boldsymbol{\tau}\|_{\alpha^*}.$$

If $\phi(\mathbf{x}) \geq 0$, it can be viewed as a weight function. Certainly, the exact value of $C(\alpha^*, \phi, \Omega)$ may be difficult to find. Nevertheless, each $\boldsymbol{\tau} \in \mathcal{Q}_\phi$ yields a computable majorant of this constant, which can be used in quantitative estimates. In particular, for $\phi = 1$ we have an upper bound of the mean value

$$|\{\!\!\{v\}\!\!\}_\Omega| \leq \frac{1}{|\Omega|}\|\boldsymbol{\tau}\|_{\alpha^*}\|\nabla v\|_\alpha. \tag{1.61}$$

Using (1.61) with $\alpha = 2$ and the identity

$$\|v\|^2 = \|v - \{\!\!\{v\}\!\!\}_\Omega\|^2 + |\Omega|\{\!\!\{v\}\!\!\}_\Omega^2, \tag{1.62}$$

we find that

$$\|v\| \leq C_{\boldsymbol{\tau}}\|\nabla v\|, \quad \text{where } C_{\boldsymbol{\tau}} := \left(C_{\mathrm{P}}^2(\Omega) + |\Omega|^{-1}\|\boldsymbol{\tau}\|^2\right)^{1/2}. \tag{1.63}$$

If the Poincaré constant C_{P} (or a majorant of it) is known, then (1.63) easily yields computable majorants of the Friedrichs constant for problems with mixed boundary conditions defined on Γ_1 and Γ_2. The condition $\operatorname{div}\boldsymbol{\tau} = 1$ (contained in the definition of $\mathcal{Q}_\phi$) can be weakened and replaced by $\{\!\!\{\operatorname{div}\boldsymbol{\tau}\}\!\!\}_\Omega = 1$. Indeed, let $\varrho(\boldsymbol{\tau}) = \operatorname{div}\boldsymbol{\tau} - 1$. Since

$$|\Omega|\,|\{\!\!\{v\}\!\!\}_\Omega| \leq \left|\int_\Omega v\varrho(\boldsymbol{\tau})\,d\mathbf{x}\right| + \left|\int_\Omega \boldsymbol{\tau}\cdot\nabla v\,d\mathbf{x}\right|,$$

we obtain

$$|\{\!\!\{v\}\!\!\}_\Omega| \leq \frac{1}{|\Omega|}(C_{\mathrm{P}}(\Omega)\|\varrho(\boldsymbol{\tau})\| + \|\boldsymbol{\tau}\|)\|\nabla v\|, \tag{1.64}$$

$$\|v\| \leq \widetilde{C}_{\boldsymbol{\tau}}\|\nabla v\|, \quad \widetilde{C}_{\boldsymbol{\tau}}^2 = C_{\mathrm{P}}^2(\Omega) + \frac{(C_{\mathrm{P}}(\Omega)\|\varrho(\boldsymbol{\tau})\| + \|\boldsymbol{\tau}\|)^2}{|\Omega|}. \tag{1.65}$$

1.3.2 Constants in Poincaré-type inequalities

The same method allows us to deduce estimates of the constants in (1.39) and (1.40). Theorem below presents an upper bound of the constant $C_1(\Omega, \Gamma)$.

Theorem 1.3.1. [261] *Suppose Γ has a positive surface measure. Then for any $v \in \widetilde{H}^1(\Omega, \Gamma)$,*

$$C_1^2(\Omega, \Gamma) \leq C_{\mathrm{P}}^2(\Omega) + \frac{|\Omega|}{|\Gamma|^2} \inf_{\substack{\boldsymbol{\tau}\in\mathcal{Q}_\Gamma(\Omega),\\ \beta>0}} E(\boldsymbol{\tau}, \beta), \tag{1.66}$$

where

$$E(\boldsymbol{\tau}, \beta) = (1+\beta)\|\boldsymbol{\tau}\|^2 + \frac{1+\beta}{\beta}C_P^2(\Omega)\left\|\operatorname{div}\boldsymbol{\tau} - \frac{|\Gamma|}{|\Omega|}\right\|^2,$$

and

$$\mathcal{Q}_\Gamma(\Omega) := \{\boldsymbol{\tau} \in H(\Omega, \mathrm{div}) \mid \boldsymbol{\tau} \cdot \mathbf{n} = 1 \quad \text{on } \Gamma, \quad \boldsymbol{\tau} \cdot \mathbf{n} = 0 \quad \text{on } \partial\Omega \setminus \Gamma\}.$$

Proof. Notice that for any $\boldsymbol{\tau} \in \mathcal{Q}_\Gamma(\Omega)$ and $v \in \widetilde{H}^1(\Omega, \Gamma)$

$$\ell_{\boldsymbol{\tau}}(v) = \int\limits_\Gamma v \, ds = 0 \quad \text{and} \quad \{\!\!\{\mathrm{div}\, \boldsymbol{\tau}\}\!\!\}_\Omega = \frac{|\Gamma|}{|\Omega|}.$$

Therefore,

$$|\Gamma| \, |\{\!\!\{v\}\!\!\}_\Omega| = \left| \int\limits_\Omega \left((\boldsymbol{\tau} \cdot \nabla v + \left(\mathrm{div}\, \boldsymbol{\tau} - \frac{|\Gamma|}{|\Omega|} \right) v \right) d\mathbf{x} \right|.$$

Estimating the terms in the right-hand side, we obtain

$$|\{\!\!\{v\}\!\!\}_\Omega| \le \frac{\|\boldsymbol{\tau}\| + C_{\mathrm{P}}(\Omega) \| \,\mathrm{div}\, \boldsymbol{\tau} - \{\!\!\{\mathrm{div}\, \boldsymbol{\tau}\}\!\!\}_\Omega \|}{|\Gamma|} \|\nabla v\|. \tag{1.67}$$

Now (1.62) and (1.67) show that for any $\beta > 0$,

$$\|v\|^2 \le C_{\mathrm{P}}^2(\Omega) \|\nabla v\|^2 + \frac{|\Omega|}{|\Gamma|^2} E(\boldsymbol{\tau}, \beta) \|\nabla v\|^2.$$

This inequality implies (1.66). □

Remark 1.3.2. *If* $\boldsymbol{\tau} \in \mathcal{Q}_\Gamma(\Omega)$ *is selected such that* $\mathrm{div}\, \boldsymbol{\tau} = \frac{|\Gamma|}{|\Omega|}$, *then the estimate* (1.66) *has the simplified form*

$$C_1^2(\Omega, \Gamma) \le C_{\mathrm{P}}^2(\Omega) + \frac{|\Omega|}{|\Gamma|^2} \inf_{\boldsymbol{\tau} \in \mathcal{Q}_\Gamma(\Omega)} \|\boldsymbol{\tau}\|^2. \tag{1.68}$$

Different $\boldsymbol{\tau}$ *yield different upper bounds of the constant. The best* $\boldsymbol{\tau}$ *in* (1.68) *is defined as* $\boldsymbol{\tau}_\dagger = \nabla u_\dagger$, *where* $u_\dagger$ *solves the auxiliary Neumann problem*

$$\Delta u_\dagger = \frac{|\Gamma|}{|\Omega|} \quad \text{in } \Omega, \qquad \nabla u_\dagger \cdot \mathbf{n} = g \text{ on } \partial\Omega,$$

where $g = 0$ *on* $\partial\Omega \setminus \Gamma$ *and* $g = 1$ *on* Γ.

Example 1.3.3. *We compare approximate values of the constant* $C_1(\Omega, \Gamma)$ *computed with the help of the above presented method with the exact ones (if they are known). Let*

$$\Omega = \square_h := (0, h) \times (0, 1) \quad \text{and} \quad \Gamma = \{x_1 = 0, x_2 \in [0, 1]\}.$$

By (1.41) *we find that* $C_1(\Omega,\Gamma) = \max\{2h,1\}\frac{1}{\pi}$. *Set* $\boldsymbol{\tau} = \{-1+\frac{x_1}{h},0\}$. *Since* $\|\boldsymbol{\tau}\|^2 = \frac{h}{3}$ *and* $\frac{|\Omega|}{|\Gamma|^2} = h$, *we use* (1.68) *and find that*

$$C_1^2(\square_h,\Gamma) \le C_{\mathrm{P}}^2(\square_h) + \frac{h^2}{3} =: \overline{C}_1(\square_h,\Gamma)\,. \tag{1.69}$$

Here $C_{\mathrm{P}}^2(\square_h) = \frac{1}{\pi^2}\max\{h^2,1\}$. *The ratio* $\frac{\overline{C}_1(\square_h,\Gamma)}{C_1(\Omega,\Gamma)}$ *changes from* 1 *to* 1.35 *if* $h \in (0,0.5]$, *from* 1.35 *to* 1.04 *in the interval* $[0.5,1]$, *and it is close to* 1.04 *for* $h>1$.

Example 1.3.4. *By* (1.69) *we also obtain estimates for simplices. Let*

$$\Omega = \triangle_h := \mathrm{Conv}\{(0,0),(h,0),(0,1)\} \text{ and } \Gamma = \{x_1 = 0, x_2 \in [0,1]\}.$$

In this case, $|\Omega| = \frac{h}{2}$. *Set* $\boldsymbol{\tau} = \{-1+\frac{x_1}{h},\frac{x_2}{h}\}$. *Then,*

$$\|\boldsymbol{\tau}\|^2 = \frac{h}{6}\left(1+\frac{1}{4}+\frac{1}{4}+\frac{1}{2h^2}\right) = h\left(\frac{1}{4}+\frac{1}{12h^2}\right)$$

and

$$C_1^2(\Omega,\Gamma) \le C_{\mathrm{P}}^2(\triangle_h) + \frac{h^2}{8} + \frac{1}{24} =: \overline{C}_1(\triangle_h,\Gamma). \tag{1.70}$$

In [174], *it was shown that* $C_{\mathrm{P}}(\triangle_h) \le \frac{\mathrm{diam}\triangle_h}{J_1}$, *where* $J_1 = 3.8317$ *is the first root of the Bessel function* J_1. *Therefore, we obtain an upper bound of the constant in the form*

$$\overline{C}_1^2(\triangle_h,\Gamma) = \frac{1+h^2}{J_1^2} + \frac{3h^2+1}{24} = \gamma h^2 + \delta, \tag{1.71}$$

where $\gamma = \frac{1}{J_1^2}+\frac{1}{8} \approx 0.1931$ *and* $\delta = \frac{1}{J_1^2}+\frac{1}{24} \approx 0.1098$. *If* $h=1$, *then we find that* $\overline{C}_1(\triangle_h,\Gamma) = 0.5504$ *(the exact constant is equal to* 0.4929, *see* [209]*). Hence we see that the above simple choice of* $\boldsymbol{\tau}$ *generates quite realistic bounds of the constant.*

Similar arguments can be applied to the simplex

$$\triangle_{a,b} := \mathrm{Conv}\{(0,0),(1,0),(a,b)\},$$

where $a \in [0,1]$ *and* $\Gamma = \{0 \le x_1 \le 1,\ x_2 = 0\}$. *In this case,*

$$C_1^2(\triangle_{a,b},\Gamma) \le \frac{\mathrm{diam}^2(\triangle_{a,b})}{J_1^2} + \frac{a^2-a+b^2}{8} + \frac{1}{24}. \tag{1.72}$$

If $a=b=\frac{1}{2}$, *then* $C_1(\triangle_{\frac{1}{2},\frac{1}{2}},\Gamma) \le 0.3314$ *(the exact constant is* 0.2465*).*

Consider the reference simplex in $\mathbb{R}^3$*:*

$$\triangle_{1,1,1} := \mathrm{Conv}\{(0,0,0),(1,0,0),(0,1,0),(0,0,1)\}$$

and set $\Gamma = \mathrm{Conv}\{(0,0,0),(1,0,0),(0,1,0)\}$. *In this case,* $|\Omega| = \frac{1}{6}$, $|\Gamma| = \frac{1}{2}$, *and* $C_{\mathrm{P}}(\triangle_{1,1,1}) \le \frac{\sqrt{2}}{\pi}$. *We use* (1.68) *with* $\boldsymbol{\tau} = \{x_1,x_2,-1+x_3\}$ *and find that*

$$\overline{C}_1^2(\triangle_{1,1,1},\Gamma) \le \frac{2}{\pi^2} + \frac{4}{45}.$$

Using affine-equivalent coordinate transformations we can deduce guaranteed bounds of the constants for various nondegenerate simplices in $\mathbb{R}^2$ *and* $\mathbb{R}^3$ *(see* [190]*).*

1.3.3 Constants in trace-type inequalities

Next we discuss briefly estimates of the constant $C_2(\Omega,\Gamma)$, where Γ is a part of $\partial\Omega$ and $v \in \widetilde{H}^1(\Omega,\Gamma)$ (see Sect. 1.2.3). If the constant $C_1(\Omega,\Gamma)$ has been defined, then an upper bound of $C_2(\Omega,\Gamma)$ follows from the integral identity

$$\int_\Gamma v^2\,ds = \int_\Omega (v^2 \operatorname{div}\boldsymbol{\tau} + \boldsymbol{\tau}\cdot\nabla(v^2)\,d\mathbf{x},$$

where $\boldsymbol{\tau}$ is selected so that $\boldsymbol{\tau}\cdot\mathbf{n} = 1$ on Γ, $\boldsymbol{\tau} \in L^\infty(\Omega,\mathbb{R}^d)$, and $\operatorname{div}\boldsymbol{\tau} \in L^\infty(\Omega)$. We have

$$\|v\|^2_\Gamma \;\le\; \|\operatorname{div}\boldsymbol{\tau}\|_{\infty,\Omega} C_1^2(\Omega,\Gamma)\|\nabla v\|^2_\Omega + \|\boldsymbol{\tau}\|_{\infty,\Omega}\|\nabla(v^2)\|_\Omega.$$

This inequality shows that $C_2(\Omega,\Gamma) \le \overline{C}_2(\Omega,\Gamma)$, where

$$\overline{C}_2^2(\Omega,\Gamma) := \|\operatorname{div}\boldsymbol{\tau}\|_{\infty,\Omega} C_1^2(\Omega,\Gamma) + 2C_1(\Omega,\Gamma)\|\boldsymbol{\tau}\|_{\infty,\Omega}. \tag{1.73}$$

Now we can obtain a computable bound of the constant in the trace estimate. Let $v \in H^1(\Omega)$ and $\boldsymbol{\tau} \in \mathcal{Q}_\Gamma$. Then

$$|\Gamma|\,|\{\!\!\{v\}\!\!\}_\Gamma| = \int_\Omega (v \operatorname{div}\boldsymbol{\tau} + \nabla v\cdot\boldsymbol{\tau})d\mathbf{x} \le (\|v\|^2_\Omega + \|\nabla v\|^2_\Omega)^{1/2}\|\boldsymbol{\tau}\|_{\operatorname{div},\Omega}.$$

Since

$$\|v\|^2_\Gamma = \|v - \{\!\!\{v\}\!\!\}_\Gamma\|^2 + |\Gamma||\{\!\!\{v\}\!\!\}_\Gamma|^2 \le C_2^2(\Omega,\Gamma)\|\nabla v\|^2_\Omega + \|v\|^2_{1,2,\Omega}\|\boldsymbol{\tau}\|^2_{\operatorname{div},\Omega},$$

we conclude that

$$\|v\|_\Gamma \le C_{\mathrm{tr}}(\Omega,\Gamma)\|v\|_{1,2,\Omega},$$

where $C^2_{\mathrm{tr}}(\Omega,\Gamma) = C_2^2(\Omega,\Gamma) + \|\boldsymbol{\tau}\|^2_{\operatorname{div},\Omega}$.

1.3.4 Estimates of constants based on domain decomposition

The estimate (1.66) yields easily computable bounds of the constant for domains with complicated boundaries if we combine it with domain decomposition. In the simplest case, Ω is decomposed into two non-overlapping domains and $\Gamma \subset \partial\Omega\cap\partial\Omega_1$ (typical examples are depicted in Fig. 1.3.1). We define $\boldsymbol{\tau}$ such that $\operatorname{div}\boldsymbol{\tau} = c \in \mathbb{R}$ in Ω_1, $\boldsymbol{\tau} = 0$ in Ω_2 and $\boldsymbol{\tau}\cdot\mathbf{n} = 0$ on $\partial\Omega_1\setminus\Gamma$. If in addition $\boldsymbol{\tau}\cdot\mathbf{n} = 1$ on Γ, then $\boldsymbol{\tau} \in \mathcal{Q}_\Gamma(\Omega)$.

Since $c = |\Gamma|/|\Omega_1|$ and $\{\!\!\{\operatorname{div}\boldsymbol{\tau}\}\!\!\}_\Omega = |\Gamma|/|\Omega|$, we find that

$$\|\operatorname{div}\boldsymbol{\tau} - \frac{|\Gamma|}{|\Omega|}\|^2_\Omega = \frac{|\Gamma|^2|\Omega_2|}{|\Omega||\Omega_1|}.$$

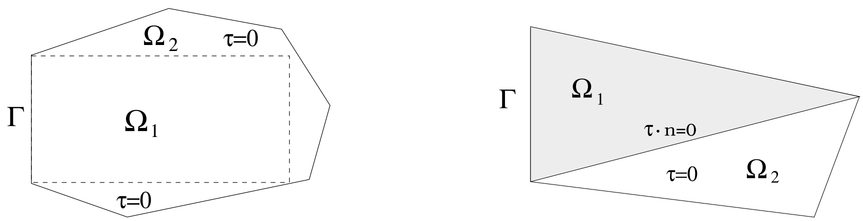

Figure 1.3.1. Decomposition of Ω into non-overlapping subdomains.

Then (1.66) yields the estimate

$$C_1^2(\Omega,\Gamma) \le C_{\mathrm{P}}^2(\Omega) + \left(\frac{|\Omega|^{1/2}}{|\Gamma|} \|\boldsymbol{\tau}\|_{\Omega_1} + C_{\mathrm{P}}(\Omega) \frac{|\Omega_2|^{1/2}}{|\Omega_1|^{1/2}} \right)^2. \tag{1.74}$$

If $|\Omega_2| = 0$, then (1.74) reduces to (1.68).

In particular, if $\Omega_1 = \square_h$ and $|\Gamma| = 1$ (see Fig. 1.3.1, left), then using (1.69) we obtain

$$C_1^2(\Omega,\Gamma) \le C_{\mathrm{P}}^2(\Omega) + \left(|\Omega|^{1/2} \sqrt{\frac{h}{3}} + C_{\mathrm{P}}(\Omega) \frac{|\Omega_2|^{1/2}}{\sqrt{h}} \right)^2. \tag{1.75}$$

If $\Omega_1 = \triangle_{a,b}$ (i.e., b is the height of the triangle) and $|\Gamma| = 1$ (Fig. 1.3.1, right), then we can take the same $\boldsymbol{\tau}$ as for the triangle $\triangle_{a,b}$. Then $\frac{|\Omega|}{|\Gamma|^2}\|\boldsymbol{\tau}\|^2_{\Omega_1}$ is defined by the last two terms of (1.72) and we can use (1.74) with $\overline{C}_{\mathrm{P}}(\Omega) = \frac{(\operatorname{diam}\Omega)}{\pi}$.

More complicated (e.g., multi connected domains) can be decomposed into a larger number of subdomains. Then estimates of $C_1^2(\Omega,\Gamma)$ can be deduced by obvious generalisations of the method discussed above. However, using (1.74), (1.75), and other similar estimates requires a computable bound of the constant $C_{\mathrm{P}}(\Omega)$. This question is considered next.

Assume that Ω can be divided into N disjoint subdomains Ω_i such that $\overline{\Omega} = \bigcup_{i=1}^N \overline{\Omega}_i$ and the constants $C_{\mathrm{P}}(\Omega_i)$ associated with the subdomains Ω_i are known (e.g., if all the subdomains are convex, then we can use the estimate (1.32)). We wish to find a computable majorant of $C_{\mathrm{P}}(\Omega)$ using these known constants.

Introduce the set of vector-valued functions $\{\mathbf{y}^{(1)}, \mathbf{y}^{(2)}, \dots, \mathbf{y}^{(N-1)}\}$ such that

$$\mathbf{y}^{(i)} \in H(\Omega, \operatorname{div}), \quad \mathbf{y}^{(i)} \cdot \mathbf{n} = 0 \quad \text{on } \Gamma = \partial\Omega. \tag{1.76}$$

Let $\omega_i = \operatorname{supp}(\mathbf{y}^{(i)})$, $\overline{\Omega} = \bigcup_{i=1}^{N-1} \omega_i$ and

$$\sum_{j=1}^{N-1} \|w\|^2_{\omega_i} \le C_\omega \|w\|^2_\Omega \quad \forall w \in L^2(\Omega), \tag{1.77}$$

where C_ω is a positive constant (it depends on the maximal number of intersections between different sets ω_i). One more requirement is that the matrix $\mathsf{B} := \{\beta_{ij}\}_{i,j=1}^{N}$, where $\beta_{ij} := \{\!|\operatorname{div}\mathbf{y}^{(i)}|\!\}_{\Omega_j}$, $\beta_{Nj} = 1$ for $i = 1,\ldots,N-1$, $j = 1,\ldots,N$, is nondegenerate, i.e.,

$$\det\mathsf{B} \neq 0. \tag{1.78}$$

Remark 1.3.5. *It is not difficult to show that functions $\mathbf{y}^{(i)}$ with the required properties exist. For example, we can set $\mathbf{y}^{(i)} = \nabla u_i$, where u_i is the solution of the problem*

$$\Delta u_i = 1 \quad in\ \Omega_i, \qquad \Delta u_i = -\frac{|\Omega_i|}{|\Omega_N|} \quad in\ \Omega_N,$$
$$\Delta u_i = 0 \quad in\ \Omega\setminus(\Omega_i\cap\Omega_N), \qquad \frac{\partial u_i}{\partial n} = 0 \quad on\ \Gamma.$$

This Neumann boundary-value problem is solvable for any $i = 1,2,\ldots,N-1$ and the corresponding matrix B has the entries $\beta_{ii} = 1$, $\beta_{iN} = -\frac{|\Omega_i|}{|\Omega_N|}$, $i = 1,\ldots,N-1$, $\beta_{ij} = 0$ if $i \neq j$ and $j \neq N$, $i = 1,\ldots,N-1, j = 1,\ldots,N$, $\beta_{Nj} = 1$, $j = 1,\ldots,N$. Since

$$\det\mathsf{B} = 1 + \sum_{i=1}^{N-1}\frac{|\Omega_i|}{|\Omega_N|} = \frac{|\Omega|}{|\Omega_N|} > 0,$$

we see that (1.78) holds. Certainly the above example has mainly a theoretical meaning and in a particular practical example the functions $\mathbf{y}^{(i)}$ can be constructed in a simpler way without solving auxiliary boundary value problems (see [257]).

Theorem 1.3.6 ([257]). *Let $\mathbf{y}^{(i)}$ satisfy the conditions (1.76)–(1.78) and $\boldsymbol{\alpha} \in \mathbb{R}^{N-1}$ be a vector with positive components α_i. Then, the following estimate holds*

$$C_{\mathrm{P}}^2(\Omega) \leq \max_{1\leq i\leq N} C_{\mathrm{P}}^2(\Omega_i) + \lambda_{\oplus}(\mathsf{D})\left(\sum_{i=1}^{N-1}(1+\alpha_i)E_i^2 + \kappa(\boldsymbol{\alpha},\mathbf{y}),\right) \tag{1.79}$$

where $\mathsf{D} = \left(\mathsf{B}^{-1}\right)^T \Upsilon\,\mathsf{B}^{-1}$, Υ is a diagonal matrix with entries $1/|\Omega_i|$, $i = 1,2,\ldots,N-1$,

$$E_i^2 = \sum_{j=1}^{N} C_{\mathrm{P}}^2(\Omega_j)\|\operatorname{div}\mathbf{y}^{(i)} - \beta_{ij}\|_{\Omega_j}^2,$$

and

$$\kappa(\boldsymbol{\alpha},\mathbf{y}) = C_\omega \max_{1\leq i\leq N}\left\{\left(1+\alpha_i^{-1}\right)\|\mathbf{y}^{(i)}\|^2\right\}.$$

Corollary 1.3.7. *If the functions* $\mathbf{y}^{(i)}$ *satisfy the condition* $\operatorname{div} \mathbf{y}^{(i)} = \text{const}$ *on any* Ω_j, $j = 1, 2, \ldots, N$ *(such vector fields can be constructed with the help of the Raviart–Thomas approximations* [245]*), then the majorant has the following simplified form:*

$$C_{\mathrm{P}}^2(\Omega) \leq \max_{1 \leq i \leq N} C_P^2(\Omega_i) + \lambda_{\oplus}(\mathsf{D}) C_\omega \sum_{j=1}^{N-1} \|\mathbf{y}^{(i)}\|^2.$$

Finally, we note that estimates of the constants $C_1(\Omega, \Gamma)$ and $C_2(\Omega, \Gamma)$ have been used in a posteriori error estimation methods for elliptic and parabolic problems (see [259, 189, 191]) and in special interpolation methods for polygonal domains (see [261, 263]). In these publications, the reader will find explicit bounds of the constants for a wide collection of domains.