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Introduction

In the following we give a brief outline of the book. For simplicity, we call a 1-dimensional compact connected complex manifold a curve. Curves are classified by their genus, and a curve of genus 0 is a projective line $\mathbb{P}^1$, and a curve of genus 1 is an elliptic curve. There exist g linearly independent holomorphic 1-forms on any curve of genus g. By taking period integrals of them we associate a g-dimensional abelian variety (a projective g-dimensional complex torus) called the Jacobian variety, and the Torelli theorem for curves claims that if their Jacobian varieties are isomorphic then the original curves are isomorphic. There exists a unique non-zero holomorphic 1-form on an elliptic curve up to a constant; on the other hand, any $K3$ surface has a unique non-zero holomorphic 2-form up to a constant. In this sense, $K3$ surfaces can be seen as a 2-dimensional generalization of elliptic curves. An elliptic curve can be realized as a cubic curve in a projective plane $\mathbb{P}^2$ by Weierstrass's $\wp$-function. On the other hand, a non-singular quartic surface in $\mathbb{P}^3$ is an example of a $K3$ surface. In the 19th century, E. Kummer discovered a $K3$ surface called the Kummer quartic surface. A Kummer quartic surface is realized as the quotient surface of the Jacobian of a curve of genus 2 and has 16 rational double points of type A_1. They form a beautiful microcosm with a line geometry in $\mathbb{P}^3$, but also are important in a proof of the Torelli-type theorem. At the present time a Kummer surface means the minimal model of the quotient surface of a 2-dimensional complex torus by the (-1)-multiplication. The set of isomorphism classes of Kummer surfaces has 4-dimensional parameters, but that of Kummer quartic surfaces has only 3-dimensional parameters. A difference from the case of curves is the existence of non-projective surfaces. For example, the existence of $K3$ surfaces not realized as quartic surfaces results from the following argument. Let V be the vector space of homogeneous polynomials of degree 4 in 4 variables. By counting monomials we know that V has dimension 35. Each point in the projective space $\mathbb{P}(V)$ defines a quartic surface and the set of isomorphism classes of quartic surfaces has $34 - \dim \mathrm{PGL}(4, \mathbb{C}) = 19$ parameters by considering the action of projective transformations. On the other hand, the isomorphism classes of all $K3$ surfaces have 20-dimensional parameters by deformation theory. Roughly speaking, the set of isomorphism classes of $K3$ surfaces is a 20-dimensional connected complex manifold in which there are countably many 19-dimensional submanifolds, each of which is the set of polarized $K3$ surfaces parametrized by an even positive integer called the degree of polarization. For example, a non-singular quartic surface has a polarization of degree 4. In the case of complex tori, they can be constructed concretely as the quotient of a complex vector space by a discrete subgroup, but it is difficult to construct a general projective $K3$ surface. This causes a difficulty in studying $K3$ surfaces uniformly.

Now we briefly recall the theory of periods of elliptic curves to understand the case of $K3$ surfaces. We denote by $\mathrm{Im}(z)$ the imaginary part of a complex number z. To each τ in the upper half-plane $H^+ = \{\tau \in \mathbb{C} : \mathrm{Im}(\tau) > 0\}$, we associate the subgroup $\mathbb{Z}+\mathbb{Z}\tau$ of the additive group $\mathbb{C}$ generated by $\{1, \tau\}$. The quotient group $E = \mathbb{C}/(\mathbb{Z}+\mathbb{Z}\tau)$ naturally has the structure of a 1-dimensional compact complex manifold, which is called an elliptic curve. A holomorphic 1-form dz on $\mathbb{C}$ is invariant under translation and hence induces a nowhere-vanishing holomorphic 1-form ω_E on E. We remark that ω_E is unique up to a constant. On the other hand, E is a 2-dimensional real torus $S^1 \times S^1$ and hence $H_1(E,\mathbb{Z}) \cong \mathbb{Z} \oplus \mathbb{Z}$. Now let us fix a basis $\{\gamma_1, \gamma_2\}$ of $H_1(E,\mathbb{Z})$. Then the integrals

$$\int_{\gamma_1} \omega_E, \quad \int_{\gamma_2} \omega_E$$

are linearly independent over $\mathbb{R}$ and therefore, if necessary by changing γ_1 and γ_2, we may assume

$$\mathrm{Im}\left(\int_{\gamma_1} \omega_E \Big/ \int_{\gamma_2} \omega_E\right) > 0.$$

Then by defining

$$\tau_E = \left(\int_{\gamma_1} \omega_E\right) \Big/ \left(\int_{\gamma_2} \omega_E\right)$$

we have a point τ_E in H^+. Here we remark that τ_E is independent of the choice of ω_E, that is, the constant multiplication, because we take the ratio of two integrals. On the other hand, τ_E depends on the choice of a basis $\{\gamma_1, \gamma_2\}$. In fact, for another basis $\{\gamma_1', \gamma_2'\}$, let

$$\tau_E' = \left(\int_{\gamma_1'} \omega_E\right) \Big/ \left(\int_{\gamma_2'} \omega_E\right) \in H^+$$

and let

$$\gamma_1' = a\gamma_1 + b\gamma_2, \quad \gamma_2' = c\gamma_1 + d\gamma_2 \quad (a, b, c, d \in \mathbb{Z})$$

be the change of basis; then we have

$$\tau_E' = \frac{a\tau_E + b}{c\tau_E + d}.$$

The matrix of a base change is contained in $\mathrm{GL}(2,\mathbb{Z})$, and the conditions $\mathrm{Im}(\tau_E) > 0$ and $\mathrm{Im}(\tau_E') > 0$ imply that $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}(2,\mathbb{Z})$. Thus the changing of a basis corresponds to the action of an element of $\mathrm{SL}(2,\mathbb{Z})$ on the upper half-plane H^+

by a linear fractional transformation. After all, the point τ_E in the quotient space $H^+/\mathrm{SL}(2,\mathbb{Z})$ is independent of the choice of holomorphic 1-forms and a basis of the homology group, and depends only on the isomorphism class of E. We call τ_E the period of the elliptic curve E and the upper half-plane the period domain. Thus the set of isomorphism classes of elliptic curves (called the moduli space of elliptic curves) bijectively corresponds to $H^+/\mathrm{SL}(2,\mathbb{Z})$ by sending an elliptic curve to its period. This is an outline of the period theory of elliptic curves.

Now we return to the case of $K3$ surfaces. Let X be a $K3$ surface on which there exists a unique nowhere-vanishing holomorphic 2-form ω_X up to a constant. By integrating it over the second homology group $H_2(X,\mathbb{Z})$,

$$\omega_X\colon H_2(X,\mathbb{Z}) \to \mathbb{C}, \quad \gamma \to \int_\gamma \omega_X,$$

ω_X can be considered an element in $H^2(X,\mathbb{C})$, which is the period of the $K3$ surface X. The second cohomology group $H^2(X,\mathbb{Z})$ is a free abelian group of rank 22, and together with the cup

$$\langle\ ,\ \rangle\colon H^2(X,\mathbb{Z}) \times H^2(X,\mathbb{Z}) \to H^4(X,\mathbb{Z}) \cong \mathbb{Z},$$

$H^2(X,\mathbb{Z})$ has the structure of a lattice. In this book a lattice means a pair of a free abelian group of finite rank and an integral-valued non-degenerate symmetric bilinear form on it. The period satisfies

$$\langle \omega_X, \omega_X\rangle = \int_X \omega_X \wedge \omega_X = 0, \quad \langle \omega_X, \bar{\omega}_X\rangle = \int_X \omega_X \wedge \bar{\omega}_X > 0,$$

which is called the Riemann condition. The topology of $K3$ surfaces is unique and is independent on complex structures. In particular, the isomorphism class of the lattice $H^2(X,\mathbb{Z})$ is independent on X and hence is denoted by L. Now we define

$$\Omega = \{\omega \in \mathbb{P}(L \otimes \mathbb{C}) : \langle \omega, \omega\rangle = 0,\ \langle \omega, \bar{\omega}\rangle > 0\},$$

which is called the period domain of $K3$ surfaces and corresponds to the upper half-plane of elliptic curves (here, for simplicity, we use the same symbol ω for a point in $L \otimes \mathbb{C}$ and its image in $\mathbb{P}(L \otimes \mathbb{C})$). Since L has rank 22, Ω is a 20-dimensional complex manifold. An isomorphism

$$\alpha_X\colon H^2(X,\mathbb{Z}) \to L$$

of lattices is called a marking for X and the pair (X, α_X) a marked $K3$ surface. To a marked $K3$ surface we associate a point $\alpha_X(\omega_X) \in \Omega$. By considering the projective space, this is independent of the choice of holomorphic 2-forms.

As in the case of elliptic curves, to get the period independent of the choice of α_X we need to take the quotient of Ω by the automorphism group $\mathrm{O}(L)$ of L, but the quotient space $\Omega/\mathrm{O}(L)$ has no complex structure. Therefore, we define the period only for marked $K3$ surfaces. And we can also define the period of a family of complex analytic surfaces $\pi\colon \mathcal{X} \to B$ which is a smooth deformation of a $K3$ surface. Here $\mathcal{X}$, B are complex manifolds, the fibers of π are $K3$ surfaces, and the fiber over the base point $t_0 \in B$ is the given $K3$ surface X. We may assume that B is a neighborhood or a germ at t_0. Moreover, we assume that B is contractible. Then a marking α_X of X induces a marking of every fiber simultaneously, and hence gives an associated holomorphic map

$$\lambda\colon B \to \Omega.$$

The map λ is called the period map for a family π. We have a map from the set of isomorphism classes of marked $K3$ surfaces to Ω by associating their periods, which is called the period map too. When we discuss the local isomorphism of the period map we use the former sense, and when discussing the surjectivity of the period map we use the period map in the latter sense.

Now consider two marked $K3$ surfaces whose periods coincide. Then the Torelli-type theorem for $K3$ surfaces answers the question of when the isomorphism

$$(\alpha_{X'})^{-1} \circ \alpha_X\colon H^2(X,\mathbb{Z}) \to H^2(X',\mathbb{Z}) \tag{0.1}$$

of lattices preserving the classes of their holomorphic 2-forms is induced from an isomorphism between X and X'. If an isomorphism is induced from an isomorphism between complex manifolds, then it preserves the classes of Kähler forms. The Torelli-type theorem claims that the converse, that is, "an isomorphism of lattices preserving holomorphic 2-forms is induced from an isomorphism of complex manifolds if and only if it preserves the classes of Kähler forms", is true. In this book we assume the fact, proved by Siu, that every $K3$ surface is Kähler. We remark that all Kähler forms form a subset of $H^2(X,\mathbb{R})$, called the Kähler cone, which is a fundamental domain for an action of some reflection group on a cone, called the positive cone of the $K3$ surface. Preserving Kähler classes is nothing but preserving the Kähler cone.

Next we discuss the periods of projective $K3$ surfaces. The pair (X,H) of a projective $K3$ surface X and a primitive ample divisor H with $H^2 = 2d$ is called a polarized $K3$ surface of degree $2d$. Here H is called primitive if the quotient module $H^2(X,\mathbb{Z})/\mathbb{Z}H$ has no torsion. It follows from lattice theory that a primitive element of L with norm $2d$ is unique up to the action of the automorphism group $\mathrm{O}(L)$ of L. Therefore, for a fixed primitive element $h \in L$ with $\langle h,h\rangle = 2d$, we can take an isomorphism $\alpha_X\colon H^2(X,\mathbb{Z}) \to L$ satisfying $\alpha_X(H) = h$. On the other hand, ω_X is perpendicular to any classes represented by curves. In particular $\langle \omega_X, H\rangle = 0$. Thus we define

$$L_{2d} = \{x \in L : \langle x, h\rangle = 0\},$$

$$\Omega_{2d} = \{\omega \in \mathbb{P}(L_{2d} \otimes \mathbb{C}) : \langle \omega, \omega\rangle = 0,\ \langle \omega, \bar{\omega}\rangle > 0\},$$

and then associate a pair (X, H, α_X) to $\alpha_X(\omega_X) \in \Omega_{2d}$. Since L_{2d} has rank 21, the set Ω_{2d} is a 19-dimensional complex manifold. The group Γ_{2d} of isomorphisms of the lattice L fixing h acts on Ω_{2d} properly discontinuously and hence the quotient Ω_{2d}/Γ_{2d} has the structure of a complex analytic space. This follows from the fact that the lattice has the signature $(2, 19)$ and hence the associated Ω_{2d} has the structure of a bounded symmetric domain (more precisely, a disjoint union of two bounded symmetric domains). We note that the upper half-plane H^+ is the simplest example of a bounded symmetric domain. We may conclude that we can define the map from the set of isomorphism classes of polarized $K3$ surfaces of degree $2d$ to Ω_{2d}/Γ_{2d}, called the period map for polarized $K3$ surfaces, and the Torelli-type theorem for polarized $K3$ surfaces claims the injectivity of this map. In this case, if the images of two polarized $K3$ surfaces under the period map coincide, then there exists an isomorphism (0.1) of lattices preserving their periods and ample classes, and in particular preserving Kähler classes, and hence the proof of the Torelli-type theorem is reduced to the case of Kähler $K3$ surfaces.

The proof of the Torelli-type theorem consists of special and peculiar arguments. First, the local isomorphism of the period map is proved by deformation theory of complex structures. On the other hand, any Kummer surface is the quotient of a complex torus, and the complex torus can be reconstructed from the period of the Kummer surface. Then the Torelli-type theorem for Kummer surfaces follows from the Torelli theorem for complex tori. Moreover, it is proved that the period points of Kummer surfaces are dense in the period domain Ω. Finally, one can prove the Torelli-type theorem for the general case by using a density argument and the Torelli-type theorem for Kummer surfaces. This is an outline of the proof.

On the other hand, the proof of the surjectivity of the period map depends on a result of the Calabi conjecture. In the case of projective $K3$ surfaces there is another proof that uses degeneration. In this book we give only a brief outline of the surjectivity of the period map.

As we will mention in some history in Remark 0.1, the Torelli-type theorem for projective $K3$ surfaces was established first, then the one for Kähler $K3$ surfaces, and it was later that the surjectivity of the period map and finally the Kählerness of $K3$ surfaces were proved. In this book we will carry out the argument under the assumption that any $K3$ surface is Kähler.

The above is the main theme of this book, but concrete geometric examples are only Kummer surfaces because we treat analytic $K3$ surfaces mainly. Therefore we will consider Enriques surfaces and plane quartic curves in the final two chapters. An Enriques surface is a non-rational algebraic surface with vanishing geometric and arithmetic genus, discovered by F. Enriques, a member of the Italian school of algebraic geometry. Any Enriques surface is algebraic and its Picard number is 10, and hence it contains many curves, and various constructions by a projective geometry are known. A $K3$ surface appears as the universal covering (the covering degree is 2) of an Enriques surface. In other words, any Enriques surface can be defined as the quotient surface of a $K3$ surface by a fixed-point-free automorphism of order 2. In the case of polarized $K3$ surfaces we fix a sublattice $\mathbb{Z}H$ of rank 1 in $H^2(X,\mathbb{Z})$, and in the case of Enriques surfaces we will fix a sublattice of rank 10, which might be a typical example of a lattice polarized $K3$ surface. In this book, as applications of the Torelli-type theorem for $K3$ surfaces, we prove the Torelli-type theorem for Enriques surfaces, and mention the automorphism groups of Enriques surfaces and various concrete constructions of Enriques surfaces. In Chapter 9 we consider, as a topic, Reye congruence associated with a line geometry which was studied in the later half of the 19th century and the beginning of the 20th century.

In Chapter 10 we give an application to non-singular plane quartic curves (quartic curves in $\mathbb{P}^2$). Plane quartics are non-hyperelliptic curves of genus 3 and their Jacobian varieties are 3-dimensional principally polarized abelian varieties. As a higher-dimensional analogue of the quotient space $H^+/\mathrm{SL}(2,\mathbb{Z})$ in the theory of elliptic curves, the quotient space $\mathfrak{H}_3/\mathrm{Sp}_6(\mathbb{Z})$ of the 3-dimensional Siegel upper half-space $\mathfrak{H}_3$ by the symplectic group $\mathrm{Sp}_6(\mathbb{Z})$ is the set of isomorphism classes of 3-dimensional principally polarized abelian varieties (called the moduli space). Since the Torelli theorem for curves implies the injectivity of the map that associates to a curve its Jacobian, and both the moduli space of plane quartic curves and $\mathfrak{H}_3/\mathrm{Sp}_6(\mathbb{Z})$ have the same dimension 6, the moduli space of plane quartics and $\mathfrak{H}_3/\mathrm{Sp}_6(\mathbb{Z})$ are birational. In this book we associate a $K3$ surface, instead of the Jacobian, with a plane quartic. To the defining equation $f(x,y,z) = 0$ of a plane quartic where f is a homogeneous polynomial of degree 4, we associate the quartic surface in $\mathbb{P}^3$ defined by $t^4 = f(x,y,z)$ where t is a new variable. The main topic in the Chapter 10 is, by using the Torelli-type theorem for $K3$ surfaces, to show that the moduli space of plane quartics is birationally isomorphic to the quotient space of a 6-dimensional complex ball by a discrete group. Moreover, we will discuss del Pezzo surfaces of degree 2 and a root system of type E_7 which are deeply related to plane quartics.

Lattice theory is necessary to discuss the Torelli-type theorem for $K3$ surfaces. First of all, we give preliminaries from lattice theory in Chapter 1. In Chapter 2 we study reflection groups and their fundamental domains. In Chapter 3 we introduce the classification of complex analytic surfaces and also the classification of singular fibers of elliptic surfaces. We give fundamental properties of $K3$ surfaces and examples (such as Kummer surfaces) of $K3$ surfaces in Chapter 4, and the Torelli theorem for 2-dimensional complex tori is proved. It will be used to prove the Torelli-type theorem for Kummer surfaces. Chapter 5 is devoted to introducing bounded symmetric domains of type IV, a higher-dimensional generalization of the upper half-plane, and then to introducing deformation theory of compact complex manifolds. This theory will be necessary for discussing the local isomorphicity of the period map of $K3$ surfaces. In Chapter 6 we give an explicit formulation of the Torelli-type theorem and its proof, and in Chapter 7 we explain the surjectivity of the period map. In Chapter 8 we give a couple of applications of the Torelli-type theorem to automorphisms of $K3$ surfaces. In Chapter 9 we introduce periods of Enriques surfaces, automorphism groups, and concrete examples. Chapter 10 is devoted to introducing plane quartic curves and related del Pezzo surfaces, and then giving a description of the moduli space of plane quartics as a complex ball quotient.

For the Torelli-type theorem for $K3$ surfaces, in addition to the original papers due to Piatetskii-Shapiro, Shafarevich [PS] and Burns, Rapoport [BR], we refer mostly to two books: the seminar note in French edited by Beauville [Be3] and the book by Barth, Hulek, Peters, Van de Ven [BHPV]. The references for algebraic and complex analytic surfaces are the articles Shafarevich [Sh], Kodaira [Kod1], [Kod2], Morrow, Kodaira [MK] and Beauville [Be1], and for the Torelli-type theorem for Enriques surfaces Namikawa [Na2]. The references are not complete and are kept to a necessary minimum. Of course this book does not cover all research on $K3$ surfaces. Topics not mentioned in this book include moduli spaces of vector bundles on a $K3$ surface and the Fourier–Mukai transform, Kähler symplectic manifolds which are higher-dimensional analogues of $K3$ surfaces, the case of positive characteristic and application to complex dynamical systems.

Remark 0.1. We summarize some history concerning the Torelli-type theorem for $K3$ surfaces. Weil [We] invented the name $K3$ surface, and thus $K3$ resulted from the initials of the three mathematicians Kummer, mentioned above, E. Kähler, and K. Kodaira, as well as from the mountain K2 located in Karakoram range, the second-highest mountain in the world (8611 m), which was unclimbed at that time (Weil's original is "ainsi nommées en l'honneur de Kummer, Kähler, Kodaira et de la belle montagne K2 au Cashemire"). Weil, together with A. Andreotti, proposed periods of $K3$ surfaces. Kodaira extended the classification of algebraic surfaces due to the Italian school to the case of complex analytic surfaces, and then established the local Torelli theorem for $K3$ surfaces (however, in [Kod2] giving the proof of this theorem, Kodaira mentioned that the local Torelli theorem is due to Andreotti and Weil). Moreover, Kodaira showed the density of periods of $K3$ surfaces with the structure of elliptic fibration in the period domain of $K3$ surfaces, and, as its application, he proved that any $K3$ surfaces are deformation equivalent and, in particular, all $K3$ surfaces are diffeomorphic.

Under the situation above, Piatetskii-Shapiro, Shafarevich [PS] had succeeded in proving the Torelli-type theorem for projective $K3$ surfaces. This was around 1970. Right after that, Burns, Rapoport [BR] succeeded in proving the Torelli-type theorem for Kähler $K3$ surfaces, not just projective ones. However, it remained open whether all $K3$ surfaces are Kähler or not. On the other hand, the surjectivity of the period map was a big remaining problem. In the middle of 1970, Horikawa [Ho1] and Shah [Sha] proved independently the surjectivity of the period map for polarized $K3$ surfaces of degree 2 by using geometric invariant theory. Right after that, Kulikov [Ku1], [Ku2], a member of the Shafarevich school, proved the surjectivity of the period map for projective $K3$ surfaces by classifying degenerations of $K3$ surfaces (right after that, Persson, Pinkham [PP] re-proved Kulikov's theorem). On the other hand, at that time Horikawa [Ho2] gave a proof of the Torelli-type theorem for Enriques surfaces. The proof of the surjectivity of the period map for the general case, not just for projective $K3$ surfaces, was given by Todorov [To] around 1980. Thus the Kählerness of $K3$ surfaces remained open and was finally solved by Siu [Si] in the first half of the 1980s.