Chapter 1
Introduction

In this introductory chapter we present in detail the main results of this monograph (Theorems 1.2.1 and 1.2.3) concerning the existence of quasiperiodic solutions of multidimensional nonlinear wave (NLW) equations with periodic boundary conditions, with a short description of the related literature. A comprehensive introduction to KAM theory for PDEs is provided in Chapter 2.

1.1 Main result and historical context

We consider autonomous NLW equations

$$u_{tt} - \Delta u + V(x)u + g(x,u) = 0, \quad x \in \mathbb{T}^d := \mathbb{R}^d/(2\pi\mathbb{Z})^d, \tag{1.1.1}$$

in any space dimension $d \geq 1$, where $V(x) \in C^\infty(\mathbb{T}^d, \mathbb{R})$ is a real-valued *multiplicative* potential and the nonlinearity $g \in C^\infty(\mathbb{T}^d \times \mathbb{R}, \mathbb{R})$ has the form

$$g(x,u) = a(x)u^3 + O\left(u^4\right) \tag{1.1.2}$$

with $a(x) \in C^\infty(\mathbb{T}^d, \mathbb{R})$. We require that the elliptic operator $-\Delta + V(x)$ be positive definite, namely there exist $\beta > 0$ such that

$$-\Delta + V(x) > \beta \, \mathrm{Id}\,. \tag{1.1.3}$$

Condition (1.1.3) is satisfied, in particular, if the potential $V(x) \geq 0$ and $V(x) \not\equiv 0$.

In this monograph we prove the existence of small-amplitude time-quasiperiodic solutions of (1.1.1). Recall that a solution $u(t,x)$ of (1.1.1) is time quasiperiodic with frequency vector $\omega \in \mathbb{R}^\nu$, $\nu \in \mathbb{N}_+ = \{1, 2, \ldots\}$, if it has the form

$$u(t,x) = U(\omega t, x)$$

where $U \colon \mathbb{T}^\nu \times \mathbb{T}^d \to \mathbb{R}$ is a C^2-function and $\omega \in \mathbb{R}^\nu$ is a nonresonant vector, namely

$$\omega \cdot \ell \neq 0, \quad \forall \ell \in \mathbb{Z}^\nu \setminus \{0\}\,.$$

If $\nu = 1$, a solution of this form is time periodic with period $2\pi/\omega$.

Small-amplitude solutions of the NLW equation (1.1.1) are approximated, at a first degree of accuracy, by solutions of the linear wave equation

$$u_{tt} - \Delta u + V(x)u = 0, \quad x \in \mathbb{T}^d\,. \tag{1.1.4}$$

In the sequel, for any $f, g \in L^2(\mathbb{T}^d, \mathbb{C})$, we denote by $(f, g)_{L^2}$ the standard L^2 inner product

$$(f, g)_{L^2} = \int_{\mathbb{T}^d} f(x)\overline{g(x)}\, dx\,.$$

There is a L^2 orthonormal basis $\{\Psi_j\}_{j\in\mathbb{N}}$, $\mathbb{N} = \{0, 1, \ldots\}$, of $L^2(\mathbb{T}^d)$ such that each Ψ_j is an eigenfunction of the Schrödinger operator $-\Delta + V(x)$. More precisely,

$$(-\Delta + V(x))\Psi_j(x) = \mu_j^2 \Psi_j(x)\,, \tag{1.1.5}$$

where the eigenvalues of $-\Delta + V(x)$

$$0 < \beta \le \mu_0^2 \le \mu_1^2 \le \cdots \le \mu_j^2 \le \cdots\,, \quad \mu_j > 0, \quad (\mu_j) \to +\infty\,, \tag{1.1.6}$$

are written in increasing order and with multiplicities. The solutions of the linear wave equation (1.1.4) are given by the linear superpositions of *normal modes* oscillations,

$$\sum_{j\in\mathbb{N}} \alpha_j \cos\left(\mu_j t + \theta_j\right) \Psi_j(x), \quad \alpha_j, \theta_j \in \mathbb{R}\,. \tag{1.1.7}$$

All of the solutions (1.1.7) of (1.1.4) are periodic, or quasiperiodic, or almost-periodic in time, with linear frequencies of oscillations μ_j, depending on the resonance properties of μ_j (which depend on the potential $V(x)$) and how many normal mode amplitudes α_j are not zero. In particular, if $\alpha_j = 0$ for any index j except a finite set $\mathbb{S}$ (tangential sites), and the frequency vector $\bar{\mu} := (\mu_j)_{j\in\mathbb{S}}$ is nonresonant, then the linear solutions (1.1.7) are quasiperiodic in time.

The main question we pose is the following:

- Do small-amplitude quasiperiodic solutions of the NLW equation (1.1.1) exist?

The main results presented in this monograph, Theorems 1.2.1 and 1.2.3, state that:

- *Small-amplitude quasiperiodic solutions* (1.1.7) *of the linear wave equation* (1.1.4)*, which are supported on finitely many indices $j \in \mathbb{S}$, persist, slightly deformed, as quasiperiodic solutions of the NLW equation* (1.1.1)*, with a frequency vector ω close to $\bar{\mu}$, for "generic" potentials $V(x)$ and coefficients $a(x)$ and "most" amplitudes $(\alpha_j)_{j\in\mathbb{S}}$.*

The potentials $V(x)$ and the coefficients $a(x)$ such that Theorem 1.2.1 holds are generic in a very strong sense; in particular they are C^∞-dense, according to Definition 1.2.2, in the set

$$\left(\mathcal{P} \cap C^\infty(\mathbb{T}^d)\right) \times C^\infty(\mathbb{T}^d)\,,$$

where $\mathcal{P} := \left\{V(x) \in H^s(\mathbb{T}^d) \colon -\Delta + V(x) > 0\right\}$ (see (1.2.36)).

Theorem 1.2.1 is a Kolmogorov–Arnold–Moser (KAM) type perturbative result. We construct recursively an embedded invariant torus $\mathbb{T}^\nu \ni \varphi \mapsto i(\varphi)$ taking values in the phase space (that we describe carefully below), supporting quasiperiodic solutions of (1.1.1) with frequency vector ω (to be determined). We employ a modified Nash–Moser iterative scheme for the search for zeros

$$\mathcal{F}(\lambda; i) = 0$$

of a nonlinear operator $\mathcal{F}$ acting on scales of Sobolev spaces of maps i, depending on a suitable parameter λ (see Chapter 6). As in a Newton scheme, the core of the problem consists in the analysis of the linearized operators

$$d_i \mathcal{F}(\lambda; \underline{i})$$

at any approximate solution $\underline{i}$ at each step of the iteration (see Section 2.1). The main task is to prove that $d_i \mathcal{F}(\lambda; \underline{i})$ admits an approximate inverse, for most values of the parameters, that satisfies suitable quantitative tame estimates in high Sobolev norms. The approximate inverse will be unbounded, i.e., it loses derivatives, due to the presence of *small divisors*. As we shall describe in detail in Section 2.5, the construction of an approximate inverse for the linearized operators obtained from (1.1.1) is a subtle problem. Major difficulties come from complicated resonance phenomena between the frequency vector ω of the expected quasiperiodic solutions and the multiple normal mode frequencies of oscillations, shifted by the nonlinearity, and the fact that the normal mode eigenfunctions $\Psi_j(x)$ are not localized close to the exponentials.

We now make a short historical introduction to KAM theory for PDEs, that we shall expand upon in Chapter 2.

As we already mentioned in the preface, in small divisors problems for PDEs, as (1.1.1), the space dimension $d = 1$ or $d \geq 2$ makes a fundamental difference due to the very different properties of the eigenvalues and eigenfunctions of the Schrödinger operator $-\Delta + V(x)$ on $\mathbb{T}^d$ for $d = 1$ and $d \geq 2$. If the space dimension $d = 1$, we also call $-\partial_{xx} + V(x)$ a Sturm–Liouville operator.

The first KAM existence results of quasiperiodic solutions were proved by Kuksin [100] (see also [102] and Wayne [126]) for 1-dimensional wave and Schrödinger (NLS) equations on the interval $x \in [0, \pi]$ with Dirichlet boundary conditions and analytic nonlinearities (see (2.3.1) and (2.3.2)). These pioneering theorems were limited to Dirichlet boundary conditions because the eigenvalues μ_j^2 of the Sturm–Liouville operator $-\partial_{xx} + V(x)$ had to be simple. Indeed the KAM scheme in [102] and [126] (see also [112]), reduces the linearized equations along the iteration to a diagonal form, with coefficients constant in time. This process requires second-order Melnikov nonresonance conditions, which concern lower bounds for differences among the linear frequencies. In these papers the potential $V(x)$ is used as a parameter to impose nonresonance conditions. Once the linearized PDEs obtained along the iteration are reduced to diagonal, constant in time form, it is easy to prove that the corresponding linear operators are invertible, for most values of the parameters, with good estimates of their inverses in high norms (with of course a loss of derivatives). We refer to Section 2.2 for a more detailed explanation of the KAM reducibility approach.

Subsequently, these results have been extended by Pöschel [113] for parameter independent nonlinear Klein–Gordon equations like (2.3.10) and by Kuksin and Pöschel [103] for NLS equations like (2.3.9). A major novelty of these papers was the use of Birkhoff normal form techniques to verify (weak) nonresonance conditions among the perturbed frequencies, tuning the amplitudes of the solutions as parameters.

In the case $x \in \mathbb{T}$, the eigenvalues of the Sturm–Liouville operator $-\partial_{xx} + V(x)$ are asymptotically double and therefore the previous second-order Melnikov nonresonance conditions are violated. In this case, the first existence results were obtained by Craig and Wayne [54] for time-periodic solutions of analytic nonlinear Klein–Gordon equations (see also [52] and [20] for completely resonant wave equations) and then extended by Bourgain [36] for time-quasiperiodic solutions. The proofs are based on a Lyapunov–Schmidt bifurcation approach and a Nash–Moser implicit function iterative scheme. The key point of these papers is that there is no diagonalization of the linearized equations at each step of the Nash–Moser iteration. The advantage is to require only minimal nonresonance conditions that are easily verified for PDEs also in presence of multiple frequencies (the second-order Melnikov nonresonance conditions are not used). On the other hand, a difficulty of this approach is that the linearized equations obtained along the iteration are variable coefficients PDEs. As a consequence it is hard to prove that the corresponding linear operators are invertible with estimates of their inverses in high norms, sufficient to imply the convergence of the iterative scheme. Relying on a "resolvent" type analysis inspired by the work of Frölich and Spencer [69] in the context of Anderson localization, Craig and Wayne [54] were able to solve this problem for time-periodic solutions in $d = 1$, and Bourgain in [36] also for quasiperiodic solutions. Key properties of this approach are:

(i) Separation properties between singular sites, namely the Fourier indices (ℓ, j) of the small divisors $|(\omega \cdot \ell)^2 - j^2| \leq C$ in the case of (NLW);

(ii) Localization of the eigenfunctions of the Sturm–Liouville operator $-\partial_{xx} + V(x)$ with respect to the exponential basis $(e^{\mathrm{i}kx})_{k \in \mathbb{Z}}$, namely that the Fourier coefficients $(\widehat{\Psi}_j)_k$ converge rapidly to zero when $||k| - j| \to \infty$. This property is always true if $d = 1$ (see, e.g., [54]).

Property (ii) implies that the matrix that represents, in the eigenfunction basis, the multiplication operator defined by an analytic (resp. Sobolev) function has an exponentially (resp. polynomially) fast decay off the diagonal. Then the separation properties (i) imply a very weak interaction between the singular sites. If the singular sites were too many, the inverse operator would be too unbounded, preventing the convergence of the iterative scheme. This approach is particularly promising in the presence of multiple normal mode frequencies and it constitutes the basis of the present monograph. We describe it in more detail in Section 2.4.

Later, Chierchia and You [48] were able to extend the KAM reducibility approach to prove the existence and stability of small-amplitude quasiperiodic solutions of 1-dimensional NLW equations on $\mathbb{T}$ with an external potential. We also mention the KAM reducibility results in Berti, Biasco and Procesi [18, 19] for 1-dimensional derivative wave equations.

When the space dimension d is greater than or equal to 2, major difficulties are:

1. The eigenvalues μ_j^2 of $-\Delta + V(x)$ in (1.1.5) may be of high multiplicity, or not sufficiently separated from each other in a suitable quantitative sense, required by the perturbation theory developed for 1-dimensional PDEs;
2. The eigenfunctions $\Psi_j(x)$ of $-\Delta + V(x)$ may be not localized with respect to the exponentials, see [64].

As discussed in the preface, if $d \geq 2$, the first KAM existence result for NLW equations has been proved for time-periodic solutions by Bourgain [37], see also the extensions in [22, 26, 76]. Concerning quasiperiodic solutions in dimension d greater than or equal to 2, the first existence result was proved by Bourgain in Chapter 20 of [42]. It deals with wave type equations of the form

$$u_{tt} - \Delta u + M_\sigma u + \varepsilon F'(u) = 0\,,$$

where $M_\sigma = \mathrm{Op}(\sigma_k)$ is a Fourier multiplier supported on finitely many sites $\mathbb{E} \subset \mathbb{Z}^d$, i.e., $\sigma_k = 0$, $\forall k \in \mathbb{Z}^d \setminus \mathbb{E}$. The σ_k, $k \in \mathbb{E}$, are used as external parameters, and F is a polynomial nonlinearity, with F' denoting the derivative of F. Note that the linear equation

$$u_{tt} - \Delta u + M_\sigma u = 0$$

is diagonal in the exponential basis $e^{\mathrm{i}k\cdot x}$, $k \in \mathbb{Z}^d$, unlike the linear wave equation (1.1.4). We also mention the paper by Wang [124] for the completely resonant NLS equation (2.3.18) and the Anderson localization result of Bourgain and Wang [45] for time-quasiperiodic random linear Schrödinger and wave equations.

As already mentioned, a major difficulty of this approach is that the linearized equations obtained along the iteration are PDEs with variable coefficients. A key property that plays a fundamental role in [42] (as well as in previous papers such as [39] for NLS) for proving estimates for the inverse of linear operators

$$\Pi_N \big((\omega \cdot \partial_\varphi)^2 - \Delta + M_\sigma + \varepsilon b(\varphi, x)\big)_{|\mathcal{H}_N}\,,$$

(see (0.0.11)) is that the matrix that represents the multiplication operator for a smooth function $b(x)$ in the exponential basis $\{e^{\mathrm{i}k\cdot x}\}$, $k \in \mathbb{Z}^d$, has a sufficiently fast off-diagonal decay. Indeed the multiplication operator is represented in Fourier space by the Toeplitz matrix $(\hat{b}_{k-k'})_{k,k'\in\mathbb{Z}^d}$, with entries given by the Fourier coefficients $\hat{b}_K$ of the function $b(x)$, constant on the diagonal $k - k' = K$. The smoother the function $b(x)$, the faster the decay of $\hat{b}_{k-k'}$ as $|k - k| \to +\infty$. We refer to Section 2.4 for more explanations on this approach.

Weaker forms of this property, as for example those required in the work of Berti, Corsi, and Procesi [27] and Berti and Procesi [33] may be sufficient for dealing with the eigenfunctions of $-\Delta$ on compact Lie groups. However, in general, the elements

$(\Psi_j, b(x)\Psi_{j'})_{L^2}$ of the matrix that represents the multiplication operator with respect to the basis of the eigenfunctions $\Psi_j(x)$ of $-\Delta + V(x)$ on $\mathbb{T}^d$ (see (1.1.5)) do not decay sufficiently fast if $d \geq 2$. This was proved by Feldman, Knörrer, and Trubowitz in [64] and it is the difficulty mentioned above in item 2. We remark that weak properties of localization have been proved by Wang [123] in $d = 2$ for potentials $V(x)$, which are trigonometric polynomials.

In the present monograph, we shall not use properties of localizations of the eigenfunctions $\Psi_j(x)$. A major reason why we are able to avoid the use of such properties is that our Nash–Moser iterative scheme requires only very weak tame estimates for the approximate inverse of the linearized operators as stated in (2.4.6), see the end of Subsection 2.4.3. Such conditions are close to the optimal ones, as a famous counterexample of Lojiaciewitz and Zehnder in [94] shows.

The properties of the exponential basis $e^{ik\cdot x}$, $k \in \mathbb{Z}^d$, also play a key role for developing the KAM perturbative diagonalization/reducibility techniques. Indeed, no reducibility results are available so far for multidimensional PDEs in presence of a multiplicative potential that is not small. Concerning higher-space dimensional PDEs, we refer to the results in Eliasson and Kuksin [61] for the NLS equation (2.3.16) with a convolution potential on $\mathbb{T}^d$ used as a parameter, Geng and You [72] and Eliasson, Grébert and Kuksin [59] for beam equations with a constant mass potential, Procesi and Procesi [117] for the completely resonant NLS equation (2.3.18), Grébert and Paturel [80] for the Klein–Gordon equation (2.3.19) on $\mathbb{S}^d$, and Grébert and Paturel [81] for multidimensional harmonic oscillators.

On the other hand, no reducibility results for NLW on $\mathbb{T}^d$ are known so far. Actually, a serious difficulty that appears is the following: the infinitely many second-order Melnikov nonresonance conditions required by the KAM-diagonalization approach are strongly violated by the linear-unperturbed frequencies of oscillations of the Klein–Gordon equation

$$u_{tt} - \Delta u + mu = 0\,,$$

see [58]. A key difference with respect to the Schrödinger equation is that the linear frequencies of the wave equations are $\sim |k|$, $k \in \mathbb{Z}^d$, while for the NLS and the beam equations, they are $\sim |k|^2$, respectively $\sim |k|^4$, and $|k|^2$, $|k|^4$ are integers. Also, for the multidimensional harmonic oscillator, the linear frequencies are, up to a translation, integer numbers. Although no reducibility results are known so far for the NLW equation, a result of "almost" reducibility for linear quasiperiodically forced Klein–Gordon equations has been presented in [57] and [58].

The existence of quasiperiodic solutions for wave equations on $\mathbb{T}^d$ with a time-quasiperiodic forcing nonlinearity of class C^∞ or C^q with q large enough,

$$u_{tt} - \Delta u + V(x)u = \varepsilon f(\omega t, x, u), \quad x \in \mathbb{T}^d\,, \tag{1.1.8}$$

has been proved in Berti and Bolle [23], extending the multiscale approach of Bourgain [42]. The forcing frequency vector ω, which in [23] is constrained to a fixed direction $\omega = \lambda\bar{\omega}$, with $\lambda \in [1/2, 3/2]$, plays the role of an external parameter.

In [27] a corresponding result has been extended for NLW equations on compact Lie groups, in [35] for Zoll manifolds, in [31] for general flat tori, and in [51] for forced Kirkhoff equations.

The existence of quasiperiodic solutions for autonomous nonlinear Klein–Gordon equations

$$u_{tt} - \Delta u + u + u^{p+1} + h.o.t. = 0, \quad p \in \mathbb{N}, \quad x \in \mathbb{T}^d, \tag{1.1.9}$$

have been recently presented by Wang [125], relying on a bifurcation analysis to study the modulation of the frequencies induced by the nonlinearity u^{p+1}, and multiscale methods of [42] for implementing a Nash–Moser iteration. The result proves the continuation of quasiperiodic solutions supported on "good" tangential sites.

Among all the works discussed above, the papers [23] and [24] on forced NLW and NLS equations are related most closely to the present monograph. The passage to prove KAM results for autonomous nonlinear wave equations with a multiplicative potential as (1.1.1) is a nontrivial task. It requires a bifurcation analysis that distinguishes the tangential directions where the major part of the oscillation of the quasiperiodic solutions takes place, and the normal ones, see the form (1.2.31) of the quasiperiodic solutions proved in Theorem 1.2.1. When the multiplicative potential $V(x)$ changes, both the tangential and the normal frequencies vary simultaneously in an intricate way (unlike the case of the convolution potential). This makes it difficult to verify the nonresonance conditions required by the Nash–Moser iteration. In particular, the choice of the parameters adopted in order to fulfill all these conditions is relevant. In this monograph we choose any finite set $\mathbb{S} \subset \mathbb{N}_+$ of tangential sites, we fix the potential $V(x)$ and the coefficient function $a(x)$ appearing in the nonlinearity (1.1.2) (in such a way that generic nonresonance and nondegeneracy conditions hold, see Theorem 1.2.3), and then we prove, in Theorem 1.2.1, the existence of quasiperiodic solutions of (1.1.1) for most values of the one dimensional internal parameter λ introduced in (1.2.24). The parameter λ amounts just to a *time rescaling* of the frequency vector ω. We also deduce a density result for the frequencies of the quasiperiodic solutions close to the unperturbed vector $\bar{\mu}$. We shall explain in more detail the choice of this parameter in Section 2.5.

1.2 Statement of the main results

In this section we state in detail the main results of this monograph, which are Theorems 1.2.1 and 1.2.3.

Under the rescaling $u \mapsto \varepsilon u$, $\varepsilon > 0$, the equation (1.1.1) is transformed into the NLW equation

$$u_{tt} - \Delta u + V(x)u + \varepsilon^2 g(\varepsilon, x, u) = 0 \tag{1.2.1}$$

with the C^∞ nonlinearity

$$g(\varepsilon, x, u) := \varepsilon^{-3} g(x, \varepsilon u) = a(x)u^3 + O\big(\varepsilon u^4\big)\,. \tag{1.2.2}$$

Recall that we list the eigenvalues $(\mu_j^2)_{j\in\mathbb{N}}$ of $-\Delta + V(x)$ in increasing order, see (1.1.6), and that we choose a corresponding L^2 orthonormal sequence of eigenfunctions $\{\Psi_j\}_{j\in\mathbb{N}}$.

We choose arbitrarily a finite set of indices $\mathbb{S} \subset \mathbb{N}$, called the tangential sites. We denote by $|\mathbb{S}| \in \mathbb{N}$ the cardinality of $\mathbb{S}$ and we list the tangential sites in increasing order, $j_1 < \cdots < j_{|\mathbb{S}|}$. We look for quasiperiodic solutions of (1.2.1) that are perturbations of normal modes oscillations supported on $j \in \mathbb{S}$. We denote by

$$\bar{\mu} := (\mu_j)_{j\in\mathbb{S}} = (\mu_{j_1}, \ldots, \mu_{j_{|\mathbb{S}|}}) \in \mathbb{R}^{|\mathbb{S}|}, \quad \mu_j > 0\,, \tag{1.2.3}$$

the frequency vector of the quasiperiodic solutions

$$\sum_{j\in\mathbb{S}} \mu_j^{-1/2}\sqrt{2\xi_j}\cos(\mu_j t)\Psi_j(x), \quad \xi_j > 0\,, \tag{1.2.4}$$

of the linear wave equation (1.1.4). The components of $\bar{\mu}$ are called the unperturbed tangential frequencies and $(\xi_j)_{j\in\mathbb{S}}$ the unperturbed actions. We shall call the indices in the complementary set $\mathbb{S}^c := \mathbb{N}\setminus\mathbb{S}$ the "normal" sites, and the corresponding μ_j, $j \in \mathbb{S}^c$, the unperturbed normal frequencies.

Since (1.2.1) is an *autonomous* PDE, the frequency vector $\omega \in \mathbb{R}^{|\mathbb{S}|}$ of its expected quasiperiodic solutions $u(\omega t, x)$ is an unknown that we introduce as an explicit parameter in the equation, looking for solutions $u(\varphi, x)$, $\varphi = (\varphi_1, \ldots, \varphi_{|\mathbb{S}|}) \in \mathbb{T}^{|\mathbb{S}|}$, of

$$(\omega\cdot\partial_\varphi)^2 u - \Delta u + V(x)u + \varepsilon^2 g(\varepsilon, x, u) = 0\,. \tag{1.2.5}$$

The frequency vector $\omega \in \mathbb{R}^{|\mathbb{S}|}$ of the expected quasiperiodic solutions of (1.2.1) will be $O(\varepsilon^2)$-close to the unperturbed tangential frequency vector $\bar{\mu}$ in (1.2.3), see (1.2.24)–(1.2.25).

Since the NLW equation (1.2.1) is time-reversible (see Appendix A), it makes sense to look for solutions of (1.2.1) that are even in t. Since (1.2.1) is autonomous, additional solutions are obtained from these even solutions by time translation. Thus we look for solutions $u(\varphi, x)$ of (1.2.5) that are *even* in φ. This induces a small simplification in the proof (see Remark 6.1.1).

In order to prove, for ε small enough, the existence of solutions of (1.2.1) close to the solutions (1.2.4) of the linear wave equation (1.1.4), we first require nonresonance conditions for the unperturbed linear frequencies μ_j, $j \in \mathbb{N}$ that will be verified by generic potentials $V(x)$, see Theorem 1.2.3.

Diophantine and first-order Melnikov nonresonance conditions. We assume that

- The tangential frequency vector $\bar{\mu}$ in (1.2.3) is Diophantine, i.e., for some constants $\gamma_0 \in (0,1)$, $\tau_0 > |\mathbb{S}| - 1$,

$$|\bar{\mu}\cdot\ell| \geq \frac{\gamma_0}{\langle\ell\rangle^{\tau_0}}, \quad \forall \ell \in \mathbb{Z}^{|\mathbb{S}|}\setminus\{0\}, \quad \langle\ell\rangle := \max\{1, |\ell|\}\,, \tag{1.2.6}$$

  where $|\ell| := \max\{|\ell_1|, \ldots, |\ell_{|\mathbb{S}|}|\}$. Note that (1.2.6) implies, in particular, that the μ_j^2, $j \in \mathbb{S}$, are simple eigenvalues of $-\Delta + V(x)$.

- The unperturbed first-order Melnikov nonresonance conditions hold:

$$|\bar{\mu} \cdot \ell + \mu_j| \geq \frac{\gamma_0}{\langle \ell \rangle^{\tau_0}}, \quad \forall \ell \in \mathbb{Z}^{|\mathbb{S}|}, \ j \in \mathbb{S}^c . \tag{1.2.7}$$

The nonresonance conditions (1.2.6) and (1.2.7) imply, in particular, that the linear equation (1.1.4) has no other quasiperiodic solutions with frequency $\bar{\mu}$, even in t, except the trivial ones (1.2.4).

In order to prove separation properties of the small divisors as required by the multiscale analysis that we perform in Chapter 5, we require, as in [23], that

- The tangential frequency vector $\bar{\mu}$ in (1.2.3) satisfies the quadratic Diophantine condition

$$\Big| n + \sum_{i,j \in \mathbb{S}, i \leq j} p_{ij} \mu_i \mu_j \Big| \geq \frac{\gamma_0}{\langle p \rangle^{\tau_0}}, \quad \forall (n, p) \in \mathbb{Z} \times \mathbb{Z}^{\frac{|\mathbb{S}|(|\mathbb{S}|+1)}{2}} \setminus \{(0,0)\} . \tag{1.2.8}$$

The nonresonance conditions (1.2.6), (1.2.7) and (1.2.8) are assumptions on the potential $V(x)$ that are "generic" in the sense of Kolmogorov measure. In [96], (1.2.6) and (1.2.7) are proved to hold for most potentials. Genericity results are stated in Theorem 1.2.3 and proved in Chapter 13.

We emphasize that throughout the monograph the constant $\gamma_0 \in (0,1)$ in (1.2.6), (1.2.7) and (1.2.8) is regarded as fixed and we shall often omit to track its dependence in the estimates.

Birkhoff matrices. We are interested in quasiperiodic solutions of (1.2.1) that bifurcate for small $\varepsilon > 0$ from a solution of the form (1.2.4) of the linear wave equation. In order to prove their existence, it is important to know precisely how the tangential and the normal frequencies change with respect to the unperturbed actions $(\xi_j)_{j \in \mathbb{S}}$, under the effect of the nonlinearity $\varepsilon^2 a(x) u^3 + O(\varepsilon^3 u^4)$. This is described in terms of the Birkhoff matrices

$$\mathscr{A} := \left(\mu_k^{-1} G_k^j \mu_j^{-1} \right)_{j,k \in \mathbb{S}}, \quad \mathscr{B} := \left(\mu_j^{-1} G_j^k \mu_k^{-1} \right)_{j \in \mathbb{S}^c, k \in \mathbb{S}}, \tag{1.2.9}$$

where, for any $j, k \in \mathbb{N}$,

$$G_k^j := G_k^j(V,a) := \begin{cases} (3/2)\big(\Psi_j^2, a(x)\Psi_k^2\big)_{L^2}, & j \neq k, \\ (3/4)\big(\Psi_j^2, a(x)\Psi_j^2\big)_{L^2}, & j = k \end{cases} \tag{1.2.10}$$

and $\Psi_j(x)$ are the eigenfunctions of $-\Delta + V(x)$ introduced in (1.1.5). Note that the matrix (G_k^j) depends on the coefficient $a(x)$ and the eigenfunctions Ψ_j, and thus on the potential $V(x)$. The $|\mathbb{S}| \times |\mathbb{S}|$ symmetric matrix $\mathscr{A}$ is called the "twist" matrix. The matrices $\mathscr{A}$, $\mathscr{B}$ describe the shift of the tangential and normal frequencies induced by the nonlinearity $a(x)u^3$ as they appear in the fourth order Birkhoff normal form of

(1.1.1) and (1.1.2). Actually, we prove in Section 6.2 that up to terms $O(\varepsilon^4)$, the tangential frequency vector ω of a small-amplitude quasiperiodic solution of (1.1.1) and (1.1.2) close to (1.2.4) is given by the action-to-frequency map

$$\xi \mapsto \bar{\mu} + \varepsilon^2 \mathscr{A} \xi, \quad \xi \in \mathbb{R}_+^{|\mathbb{S}|}, \quad \mathbb{R}_+ := (0, +\infty) . \tag{1.2.11}$$

On the other hand the perturbed normal frequencies are shifted by the matrix $\mathscr{B}$ as described in Lemma 8.3.2. We assume that

- **(Twist condition)**

$$\det \mathscr{A} \neq 0 , \tag{1.2.12}$$

and therefore the action-to-frequency map in (1.2.11) is invertible. The nondegeneracy, or twist condition (1.2.12), is satisfied for generic $V(x)$ and $a(x)$, as stated in Theorem 1.2.3 (see in particular Corollary 13.1.10 and Remark 13.1.11).

Second-order Melnikov nonresonance conditions. We also assume second-order Melnikov nonresonance conditions that concern only *finitely* many unperturbed normal frequencies. We must first introduce an important decomposition of the normal indices $j \in \mathbb{S}^c$. Note that since $\mu_j \to +\infty$, the indices $j \in \mathbb{S}^c$ such that $\mu_j - (\mathscr{B}\mathscr{A}^{-1}\bar{\mu})_j < 0$ are finitely many. Let $\mathfrak{g} \in \mathbb{R}$ be defined by

$$-\mathfrak{g} := \min \left\{ \mu_j - \left(\mathscr{B}\mathscr{A}^{-1}\bar{\mu} \right)_j, \, j \in \mathbb{S}^c \right\} . \tag{1.2.13}$$

We decompose the set of normal indices as

$$\mathbb{S}^c = \mathbb{F} \cup \mathbb{G}, \quad \mathbb{G} := \mathbb{S}^c \setminus \mathbb{F} , \tag{1.2.14}$$

where

$$\begin{aligned} \mathbb{F} &:= \left\{ j \in \mathbb{S}^c \colon \left| \mu_j - \left(\mathscr{B}\mathscr{A}^{-1}\bar{\mu} \right)_j \right| \le \mathfrak{g} \right\} , \\ \mathbb{G} &:= \left\{ j \in \mathbb{S}^c \colon \mu_j - \left(\mathscr{B}\mathscr{A}^{-1}\bar{\mu} \right)_j > \mathfrak{g} \right\} . \end{aligned} \tag{1.2.15}$$

The set $\mathbb{F}$ is always finite, and is empty if $\mathfrak{g} < 0$. The relevance of the decomposition (1.2.14) of the normal sites, concerns the variation of the normal frequencies with respect to the length of the tangential frequency vector, as we describe in (2.5.26) below, see also Lemma 8.1.1. If all the numbers $\mu_j - (\mathscr{B}\mathscr{A}^{-1}\bar{\mu})_j$, $j \in \mathbb{S}^c$, were positive, then by (2.5.26), the growth rates of the eigenvalues of the linear operators that we need to invert in the Nash–Moser iteration would all have the same sign. This would allow us to obtain measure estimates by simple arguments, as in [23] and [24], where the forced case is considered. In general $\mathfrak{g} > 0$ and we shall be able to decouple, for most values of the parameter λ, the linearized operators obtained at each step of the nonlinear Nash–Moser iteration, acting in the normal subspace $H_{\mathbb{S}}^{\perp}$, along $H_{\mathbb{F}}$ and its orthogonal $H_{\mathbb{F}}^{\perp} = H_{\mathbb{G}}$, defined in (2.5.12). We discuss the relevance of this decomposition in Section 2.5.

We assume the following

- Unperturbed "second-order Melnikov" nonresonance conditions (part 1):

$$|\bar{\mu} \cdot \ell + \mu_j - \mu_k| \geq \frac{\gamma_0}{\langle \ell \rangle^{\tau_0}}, \quad \forall (\ell, j, k) \in \mathbb{Z}^{|\mathbb{S}|} \times \mathbb{F} \times \mathbb{S}^c, \quad (\ell, j, k) \neq (0, j, j), \tag{1.2.16}$$

$$|\bar{\mu} \cdot \ell + \mu_j + \mu_k| \geq \frac{\gamma_0}{\langle \ell \rangle^{\tau_0}}, \quad \forall (\ell, j, k) \in \mathbb{Z}^{|\mathbb{S}|} \times \mathbb{F} \times \mathbb{S}^c. \tag{1.2.17}$$

Note that (1.2.16) implies, in particular, that the finitely many eigenvalues μ_j^2 of $-\Delta + V(x)$, $j \in \mathbb{F}$, are simple (clearly all the other eigenvalues μ_j^2, $j \in \mathbb{G}$, could be highly degenerate).

In order to verify a key positivity property for the variations of the restricted linearized operator with respect to λ (Lemma 10.3.8), we assume further

- Unperturbed second-order Melnikov nonresonance conditions (part 2):

$$|\bar{\mu} \cdot \ell + \mu_j - \mu_k| \geq \frac{\gamma_0}{\langle \ell \rangle^{\tau_0}}, \quad \forall (\ell, j, k) \in \mathbb{Z}^{|\mathbb{S}|} \times (\mathbb{M} \setminus \mathbb{F}) \times \mathbb{S}^c, \quad (\ell, j, k) \neq (0, j, j), \tag{1.2.18}$$

$$|\bar{\mu} \cdot \ell + \mu_j + \mu_k| \geq \frac{\gamma_0}{\langle \ell \rangle^{\tau_0}}, \quad \forall (\ell, j, k) \in \mathbb{Z}^{|\mathbb{S}|} \times (\mathbb{M} \setminus \mathbb{F}) \times \mathbb{S}^c, \tag{1.2.19}$$

  where

$$\mathbb{M} := \{ j \in \mathbb{S}^c : |j| \leq C_1 \} \tag{1.2.20}$$

  and the constant $C_1 := C_1(V, a) > 0$ is taken large enough so that $\mathbb{F} \subset \mathbb{M}$ and (8.1.8) holds.

Clearly the conditions (1.2.16)–(1.2.19) could have been written together, requiring such conditions for $j \in \mathbb{M}$ without distinguishing the cases $j \in \mathbb{F}$ and $j \in \mathbb{M} \setminus \mathbb{F}$. However, for conceptual clarity, in view of their different role in the proof, we prefer to state them separately. The above conditions (1.2.16)–(1.2.19) on the unperturbed frequencies allow us to perform one step of averaging and so to diagonalize, up to $O(\varepsilon^4)$, the normal frequencies supported on $\mathbb{M}$, see Proposition 8.3.1. This is the only step where conditions (1.2.18) and (1.2.19) play a role. Conditions (1.2.16) and (1.2.17) are also used in the splitting step of Chapter 10, see Lemma 10.3.3.

Conditions (1.2.16)–(1.2.19) depend on the potential $V(x)$ and also on the coefficient $a(x)$, because the constant C_1 in (1.2.20) (hence the set $\mathbb{M}$) depends on $\|a\|_{L^\infty}$ and $\|\mathcal{A}^{-1}\|$. Given (V_0, a_0) such that the matrix $\mathscr{A}$ defined in (1.2.9) is invertible and $s > d/2$, the set $\mathbb{M}$ can be chosen constant in some open neighborhood U of (V_0, a_0) in H^s. In U, conditions (1.2.16)–(1.2.19) are generic for $V(x)$, as it is proved in Chapter 13 (see Theorem 1.2.3).

Nondegeneracy conditions. We also require the following *finitely* many

- Nondegeneracy conditions:

$$\big(\mu_j - [\mathscr{B}\mathscr{A}^{-1}\bar{\mu}]_j\big) - \big(\mu_k - [\mathscr{B}\mathscr{A}^{-1}\bar{\mu}]_k\big) \neq 0\,, \qquad \forall j,k \in \mathbb{F},\ j \neq k\,, \tag{1.2.21}$$

$$\big(\mu_j - [\mathscr{B}\mathscr{A}^{-1}\bar{\mu}]_j\big) + \big(\mu_k - [\mathscr{B}\mathscr{A}^{-1}\bar{\mu}]_k\big) \neq 0, \quad \forall j,k \in \mathbb{F}\,, \tag{1.2.22}$$

  where $\mathscr{A}$ and $\mathscr{B}$ are the Birkhoff matrices defined in (1.2.9).

Such assumptions are similar to the nondegeneracy conditions required for the continuation of elliptic tori for finite-dimensional systems in [56, 111] and for PDEs in [17, 103, 113]. Note that the finitely many nondegeneracy conditions (1.2.21) depend on $V(x)$ and $a(x)$ and we prove in Theorem 1.2.3 that they are generic in (V,a).

Parameter. We now introduce the one-dimensional parameter that we shall use to perform the measure estimates.

In view of (1.2.11) the frequency vector ω has to belong to the cone of the "admissible" frequencies $\bar{\mu} + \varepsilon^2\mathscr{A}(\mathbb{R}_+^{|\mathbb{S}|})$, more precisely we require that ω belongs to the image

$$\mathcal{A} := \bar{\mu} + \varepsilon^2\Big\{\mathscr{A}\xi\colon \frac{1}{2} \le \xi_j \le 4, \forall j \in \mathbb{S}\Big\} \subset \mathbb{R}^{|\mathbb{S}|} \tag{1.2.23}$$

of the compact set of actions $\xi \in [1/2, 4]^{|\mathbb{S}|}$ under the approximate action-to-frequency map (1.2.11). Then, in view of the method that we shall use for the measure estimates for the linearized operator, we look for quasiperiodic solutions with frequency vector

$$\omega = \big(1+\varepsilon^2\lambda\big)\bar{\omega}_\varepsilon, \quad \lambda \in \Lambda := [-\lambda_0, \lambda_0]\,, \tag{1.2.24}$$

constrained to a fixed admissible direction

$$\bar{\omega}_\varepsilon := \bar{\mu} + \varepsilon^2\zeta, \quad \zeta \in \mathscr{A}\left([1,2]^{|\mathbb{S}|}\right). \tag{1.2.25}$$

Note that in general we can not choose $\bar{\omega}_\varepsilon = \bar{\mu}$, because $\zeta = 0$ might not belong to $\mathscr{A}([1,2]^{|\mathbb{S}|})$. We fix ζ below so that the Diophantine conditions (1.2.29) and (1.2.30) hold.

In (1.2.24) there exists $\lambda_0 > 0$ small, independent of $\varepsilon > 0$ and of $\zeta \in \mathscr{A}([1,2]^{|\mathbb{S}|})$, such that,

$$\forall \lambda \in \Lambda := [-\lambda_0, \lambda_0]\,, \quad \omega = \big(1+\varepsilon^2\lambda\big)\bar{\omega}_\varepsilon \in \mathcal{A} \tag{1.2.26}$$

are still admissible (see (1.2.23)) and, using (1.2.25),

$$\begin{aligned}&\big(1+\varepsilon^2\lambda\big)\bar{\omega}_\varepsilon = \bar{\mu} + \varepsilon^2\mathscr{A}(\xi) \quad \Longleftrightarrow \\ &\quad \xi := \xi(\lambda) = \big(1+\varepsilon^2\lambda\big)\mathscr{A}^{-1}\zeta + \lambda\mathscr{A}^{-1}\bar{\mu}\,.\end{aligned} \tag{1.2.27}$$

We shall use the one-dimensional "parameter" $\lambda \in \Lambda := [-\lambda_0, \lambda_0]$ in order to verify all the nonresonance conditions required for the frequency vector ω in the proof of Theorem 1.2.1.

For ε small and fixed, we take the vector ζ such that the direction $\bar{\omega}_\varepsilon$ in (1.2.25) still verifies Diophantine conditions like (1.2.6) and (1.2.8) with the different exponents

$$\gamma_1 := \gamma_0/2, \quad \tau_1 := 3\tau_0 + |\mathbb{S}|(|\mathbb{S}|+1) + 5 > \tau_0\,, \tag{1.2.28}$$

namely

$$|\bar{\omega}_\varepsilon \cdot \ell| \geq \frac{\gamma_1}{\langle \ell \rangle^{\tau_1}}, \quad \forall \ell \in \mathbb{Z}^{|\mathbb{S}|} \setminus \{0\}\,, \tag{1.2.29}$$

$$\left| n + \sum_{1 \leq i \leq j \leq |\mathbb{S}|} p_{ij} (\bar{\omega}_\varepsilon)_i (\bar{\omega}_\varepsilon)_j \right| \geq \frac{\gamma_1}{\langle p \rangle^{\tau_1}}, \quad \forall (n,p) \in \mathbb{Z} \times \mathbb{Z}^{\frac{|\mathbb{S}|(|\mathbb{S}|+1)}{2}} \setminus \{(0,0)\}\,. \tag{1.2.30}$$

This is possible by Lemma 3.3.1. Actually, the vector $\bar{\omega}_\varepsilon = \bar{\mu} + \varepsilon^2 \zeta$ satisfies (1.2.29) and (1.2.30) for all $\zeta \in \mathscr{A}([1,2]^{|\mathbb{S}|})$ except a small set of measure $O(\varepsilon)$. In (1.2.30), we denote, for $i = 1, \dots, |\mathbb{S}|$, the i-component $(\bar{\omega}_\varepsilon)_i = \mu_{j_i} + \varepsilon^2 \zeta_i$, where $j_1, \dots, j_{|\mathbb{S}|}$ are the tangential sites ordered according to (1.2.3).

Main result. We may now state in detail the main result of this monograph, concerning the existence of quasiperiodic solutions of the NLW equation (1.1.1).

Let $\mathcal{H}^s$ denote the standard Sobolev space $\mathcal{H}^s(\mathbb{T}^{|\mathbb{S}|} \times \mathbb{T}^d; \mathbb{R})$ of functions $u\colon \mathbb{T}^{|\mathbb{S}|} \times \mathbb{T}^d \to \mathbb{R}$, with norm $\| \ \|_s$, see (1.3.1).

Theorem 1.2.1 (Quasiperiodic solutions for the NLW equation (1.1.1)). *Let a be a function in $C^\infty(\mathbb{T}^d, \mathbb{R})$. Take V in $C^\infty(\mathbb{T}^d, \mathbb{R})$ such that $-\Delta + V(x) \geq \beta \mathrm{Id}$ for some $\beta > 0$ (see (1.1.3)) and let $\{\Psi_j\}_{j \in \mathbb{N}}$ be an L^2 basis of eigenfunctions of $-\Delta + V(x)$, with eigenvalues μ_j^2 written in increasing order and with multiplicity as in (1.1.6). Fix finitely many tangential sites $\mathbb{S} \subset \mathbb{N}$.*

Assume the following conditions:

- (i) *The unperturbed frequency vector $\bar{\mu} = (\mu_j)_{j \in \mathbb{S}} \in \mathbb{R}^{|\mathbb{S}|}$ in (1.2.3) satisfies the Diophantine conditions (1.2.6) and (1.2.8);*
- (ii) *The unperturbed first- and second-order Melnikov nonresonance conditions (1.2.7) and (1.2.16)–(1.2.19) hold;*
- (iii) *The Birkhoff matrix $\mathscr{A}$ defined in (1.2.9) satisfies the twist condition (1.2.12);*
- (iv) *The finitely many nondegeneracy conditions (1.2.21) and (1.2.22) hold.*

Finally, fix a vector

$$\bar{\omega}_\varepsilon := \bar{\mu} + \varepsilon^2 \zeta, \quad \zeta \in \mathscr{A}\big([1,2]^{|\mathbb{S}|}\big) \subset \mathbb{R}^{|\mathbb{S}|}\,,$$

such that the Diophantine conditions (1.2.29) and (1.2.30) hold.

Then the following holds:

1. *There exists a Cantor-like set* $\mathcal{G}_{\varepsilon,\zeta} \subset \Lambda$ *(with* Λ *defined as in* (1.2.26)*) with asymptotically full measure, i.e.,*

$$\left|\Lambda \setminus \mathcal{G}_{\varepsilon,\zeta}\right| \to 0 \quad as \quad \varepsilon \to 0\,;$$

2. *For all* $\lambda \in \mathcal{G}_{\varepsilon,\zeta}$*, there exists a solution* $u_{\varepsilon,\lambda} \in C^\infty(\mathbb{T}^{|\mathbb{S}|} \times \mathbb{T}^d; \mathbb{R})$ *of the equation* (1.2.5)*, of the form*

$$u_{\varepsilon,\lambda}(\varphi, x) = \sum_{j\in\mathbb{S}} \mu_j^{-1/2} \sqrt{2\xi_j} \cos(\varphi_j)\Psi_j(x) + r_\varepsilon(\varphi, x)\,, \tag{1.2.31}$$

which is even in φ*, where*

$$\omega = \big(1 + \varepsilon^2\lambda\big)\bar{\omega}_\varepsilon, \quad \bar{\omega}_\varepsilon = \bar{\mu} + \varepsilon^2\zeta\,,$$

and $\xi := \xi(\lambda) \in [1/2, 4]^{\mathbb{S}}$ *is given in* (1.2.27)*. For any* $s \ge s_0 > (|\mathbb{S}| + d)/2$*, the remainder term* r_ε *satisfies* $\|r_\varepsilon\|_s \to 0$ *as* $\varepsilon \to 0$*.*

As a consequence $\varepsilon u_{\varepsilon,\lambda}(\omega t, x)$ *is a quasiperiodic solution of the NLW equation* (1.1.1) *with frequency vector* $\omega = (1 + \varepsilon^2\lambda)\bar{\omega}_\varepsilon$*.*

Theorem 1.2.1 is a direct consequence of Theorem 6.1.2. Note that the nonresonance conditions (i), (ii) in Theorem 1.2.1 depend on the tangential sites $\mathbb{S}$ and the potential $V(x)$, whereas the nondegeneracy conditions (iii), (iv) depend on $\mathbb{S}$, $V(x)$, the choice of the basis $\{\Psi_j\}_{j\in\mathbb{N}}$ of eigenvectors of $-\Delta + V(x)$, and the coefficient $a(x)$ in the nonlinear term $g(x, u) = a(x)u^3 + O(u^4)$ in (1.1.2).

In Section 2.5 we shall provide a detailed presentation of the strategy of proof of Theorem 1.2.1. Let us now make some comments on this result.

1. **Measure estimate of** $\mathcal{G}_{\varepsilon,\zeta}$**.** The speed of convergence of $|\Lambda \setminus \mathcal{G}_{\varepsilon,\zeta}|$ to 0 does not depend on ζ. More precisely ($\gamma_0, \gamma_1, \tau_0, \tau_1$ being fixed) there is a map $\varepsilon \mapsto b(\varepsilon)$, satisfying $\lim_{\varepsilon\to 0} b(\varepsilon) = 0$, such that, for all $\zeta \in \mathscr{A}([1, 2]^{|\mathbb{S}|})$ with $\bar{\omega}_\varepsilon = \bar{\mu} + \varepsilon^2\zeta$ satisfying the Diophantine conditions (1.2.29) and (1.2.30), it follows that $|\Lambda \setminus \mathcal{G}_{\varepsilon,\zeta}| \le b(\varepsilon)$.
2. **Density.** Integrating in λ along all possible admissible directions $\bar{\omega}_\varepsilon$ in (1.2.25), we deduce the existence of quasiperiodic solutions of (1.1.1) for a set of frequency vectors ω of positive measure. More precisely, defining the convex subsets of $\mathbb{R}^{|\mathbb{S}|}$,

$$\begin{aligned} &\mathcal{C}_2 := \bar{\mu} + \mathbb{R}_+\mathcal{C}_1\,, \\ &\mathcal{C}_1 := \mathscr{A}\big([1, 2]^{|\mathbb{S}|}\big) + \Lambda\bar{\mu} := \Big\{\zeta + \lambda\bar{\mu}; \zeta \in \mathscr{A}\big([1, 2]^{|\mathbb{S}|}\big), \lambda \in \Lambda\Big\}\,, \end{aligned} \tag{1.2.32}$$

the set Ω of the frequency vectors ω of the quasiperiodic solutions of (1.1.1) provided by Theorem 1.2.1 has Lebesgue density 1 at $\bar{\mu}$ in $\mathcal{C}_2$, *i.e.*,

$$\lim_{r\to 0+} \frac{|\Omega \cap \mathcal{C}_2 \cap B(\bar{\mu}, r)|}{|\mathcal{C}_2 \cap B(\bar{\mu}, r)|} = 1\,, \tag{1.2.33}$$

where $B(\bar{\mu}, r)$ denotes the ball in $\mathbb{R}^{|\mathbb{S}|}$ of radius r centered at $\bar{\mu}$ (see the proof of (1.2.33) after Theorem 6.1.2). Moreover, we restrict ourself to $\zeta \in \mathscr{A}([1,2]^{|\mathbb{S}|})$ just to fix ideas, and we could replace this condition by $\zeta \in \mathscr{A}([r,R]^{|\mathbb{S}|})$, for any $0 < r < R$ (at the cost of stronger smallness conditions for λ_0 and ε if r is small and R is large). Therefore we could obtain a similar density result with $\mathcal{C}_2' := \bar{\mu} + \mathscr{A}(\mathbb{R}_+^{|\mathbb{S}|})$ instead of $\mathcal{C}_2$.

3. **Lipschitz dependence.** The solution $u_{\varepsilon,\lambda}$ is a Lipschitz function of $\lambda \in \mathcal{G}_{\varepsilon,\zeta}$ with values in $\mathcal{H}^s$, for any $s \geq s_0$.
4. **Regularity.** Theorem 1.2.1 also holds if the nonlinearity $g(x,u)$ and the potential $V(x)$ in (1.1.1) are of class C^q for q large enough, proving the existence of a solution $u_{\varepsilon,\lambda}$ in the Sobolev space $\mathcal{H}^{\bar{s}}$ for some finite $\bar{s} > s_0 > (|\mathbb{S}| + d)/2$ (see Remark 12.2.9).

Theorem 1.2.3 below proves that, for any choice of finitely many tangential sites $\mathbb{S} \subset \mathbb{N}$, all the nonresonance and nondegeneracy assumptions (i)–(iv) required in Theorem 1.2.1 are verified for generic potentials $V(x)$ and coefficients $a(x)$ in the nonlinear term $g(x,u) = a(x)u^3 + O(u^4)$ in (1.1.2). In order to state a precise result, we introduce the following definition.

Definition 1.2.2 (C^∞-dense open). *Given an open subset $\mathcal{U}$ of $H^s(\mathbb{T}^d)$ (resp. $H^s(\mathbb{T}^d) \times H^s(\mathbb{T}^d)$) a subset $\mathcal{V}$ of $\mathcal{U}$ is said to be C^∞-dense open in $\mathcal{U}$ if*

1. *$\mathcal{V}$ is open for the topology defined by the $H^s(\mathbb{T}^d)$-norm,*
2. *$\mathcal{V}$ is C^∞-dense in $\mathcal{U}$, in the sense that, for any $w \in \mathcal{U}$, there is a sequence $(h_n) \in C^\infty(\mathbb{T}^d)$ (resp. $C^\infty(\mathbb{T}^d) \times C^\infty(\mathbb{T}^d)$) such that $w + h_n \in \mathcal{V}$, for all $n \in \mathbb{N}$, and $h_n \to 0$ in H^r for any $r \geq 0$.*

Let $s > d/2$ and define the subset of potentials

$$\mathcal{P} := \left\{ V \in H^s(\mathbb{T}^d) \colon -\Delta + V(x) > 0 \right\} . \tag{1.2.34}$$

The set $\mathcal{P}$ is open in $H^s(\mathbb{T}^d)$ and convex, thus connected.

Given a finite subset $\mathbb{S} \subset \mathbb{N}$ of tangential sites, consider the set $\widetilde{\mathcal{G}}$ of potentials $V(x)$ and coefficients $a(x)$ such that the nonresonance and nondegeneracy conditions (i)–(iv) required in Theorem 1.2.1 hold, namely

$$\widetilde{\mathcal{G}} := \left\{ (V,a) \in \mathcal{P} \times H^s(\mathbb{T}^d) \colon \text{(i)–(iv) in Theorem 1.2.1 hold} \right\} . \tag{1.2.35}$$

Note that the nondegeneracy properties (iii), (iv) may depend on the choice (1.1.5) of the basis $\{\Psi_j\}_{j\in\mathbb{N}}$ of eigenfunctions of $-\Delta + V(x)$ (if some eigenvalues are not simple). In the above definition of $\widetilde{\mathcal{G}}$, it is understood that properties (i)–(iv) hold for some choice of the basis $\{\Psi_j\}_{j\in\mathbb{N}}$.

Given a subspace E of $L^2(\mathbb{T}^d)$, we denote by $E^\perp := E^{\perp_{L^2}}$ its orthogonal complement with respect to the L^2 scalar product.

Theorem 1.2.3 (Genericity). *Let $s > d/2$. The set*

$$\widetilde{\mathcal{G}} \cap \Big(C^\infty(\mathbb{T}^d) \times C^\infty(\mathbb{T}^d)\Big) \text{ is } C^\infty\text{-dense in } \Big(\mathcal{P} \cap C^\infty(\mathbb{T}^d)\Big) \times C^\infty(\mathbb{T}^d) \quad (1.2.36)$$

where $\mathcal{P} \subset H^s(\mathbb{T}^d)$ is the open and connected set of potentials $V(x)$ defined in (1.2.34).

More precisely, there is a C^∞-dense open subset $\mathcal{G}$ of $\mathcal{P} \times H^s(\mathbb{T}^d)$ and a $|\mathbb{S}|$-dimensional linear subspace E of $C^\infty(\mathbb{T}^d)$ such that, for all $v_2(x) \in E^\perp \cap H^s(\mathbb{T}^d)$, $a(x) \in H^s(\mathbb{T}^d)$, the Lebesgue measure (on the finite-dimensional space $E \simeq \mathbb{R}^{|\mathbb{S}|}$)

$$\Big|\{v_1 \in E \colon (v_1 + v_2, a) \in \mathcal{G} \setminus \widetilde{\mathcal{G}}\}\Big| = 0 \,. \quad (1.2.37)$$

Theorem 1.2.3 is proved in Chapter 13.

For the convenience of the reader, we provide in the next chapter a nontechnical survey of the main methods and results in KAM theory for PDEs.

1.3 Basic notation

We collect here some basic notation used throughout the monograph.

For $k = (k_1, \ldots, k_q) \in \mathbb{Z}^q$, we set

$$|k| := \max\{|k_1|, \ldots, |k_q|\} \,,$$

and, for $(\ell, j) \in \mathbb{Z}^{|\mathbb{S}|} \times \mathbb{Z}^d$,

$$\langle \ell, j \rangle := \max(|\ell|, |j|, 1) \,.$$

For $s \in \mathbb{R}$ we denote the Sobolev spaces

$$\begin{aligned} \mathcal{H}^s &:= \mathcal{H}^s(\mathbb{T}^{|\mathbb{S}|} \times \mathbb{T}^d; \mathbb{C}^r) \\ &:= \Big\{ u(\varphi, x) = \sum_{(\ell, j) \in \mathbb{Z}^{|\mathbb{S}|} \times \mathbb{Z}^d} u_{\ell, j} e^{\mathrm{i}(\ell \cdot \varphi + j \cdot x)} \colon u_{\ell, j} \in \mathbb{C}^r, \\ &\qquad \|u\|_s^2 := \sum_{(\ell, j) \in \mathbb{Z}^{|\mathbb{S}| + d}} |u_{\ell, j}|^2 \langle \ell, j \rangle^{2s} < \infty \Big\} \end{aligned} \quad (1.3.1)$$

and we also use the same notation $\mathcal{H}^s$ for the subspace of real-valued functions. Moreover we denote by $H^s := H^s_x$ the Sobolev space of functions $u(x)$ in $H^s(\mathbb{T}^d, \mathbb{C})$ and H^s_φ the Sobolev space of functions $u(\varphi)$ in $H^s(\mathbb{T}^{|\mathbb{S}|}, \mathbb{C})$. We denote by $b := |\mathbb{S}| + d$. For

$$s \geq s_0 > (|\mathbb{S}| + d)/2 \,, \quad (1.3.2)$$

we have the continuous embedding

$$\mathcal{H}^s(\mathbb{T}^{|\mathbb{S}|}\times\mathbb{T}^d)\hookrightarrow C^0(\mathbb{T}^{|\mathbb{S}|}\times\mathbb{T}^d)$$

and each $\mathcal{H}^s$ is an algebra with respect to the product of functions.

Let E be a Banach space. Given a continuous map $u\colon \mathbb{T}^{|\mathbb{S}|}\to E$, $\varphi\mapsto u(\varphi)$, we denote by $\widehat{u}(\ell)\in E$, $\ell\in\mathbb{Z}^{|\mathbb{S}|}$, its Fourier coefficients

$$\widehat{u}(\ell):=\frac{1}{(2\pi)^{|\mathbb{S}|}}\int_{\mathbb{T}^{|\mathbb{S}|}}u(\varphi)e^{-\mathrm{i}\ell\cdot\varphi}\,d\varphi\,,$$

and its average

$$\langle u\rangle:=\widehat{u}(0):=\frac{1}{(2\pi)^{|\mathbb{S}|}}\int_{\mathbb{T}^{|\mathbb{S}|}}u(\varphi)\,d\varphi\,.$$

Given a nonresonant vector $\omega\in\mathbb{R}^{|\mathbb{S}|}$, i.e., $\omega\cdot\ell\neq 0$, $\forall\ell\in\mathbb{Z}^{|\mathbb{S}|}\setminus\{0\}$, and a function $g(\varphi)\in\mathbb{R}^{|\mathbb{S}|}$ with zero average, we define the solution $h(\varphi)$ of $\omega\cdot\partial_\varphi h=g$, with zero average,

$$h(\varphi)=(\omega\cdot\partial_\varphi)^{-1}g:=\sum_{\ell\in\mathbb{Z}^{|\mathbb{S}|}\setminus\{0\}}\frac{\widehat{g}(\ell)}{\mathrm{i}\omega\cdot\ell}e^{\mathrm{i}\ell\cdot\varphi}\,. \tag{1.3.3}$$

Let E be a Banach space with norm $\|\ \|_E$. Given a function $f\colon \Lambda:=[-\lambda_0,\lambda_0]\subset\mathbb{R}\to E$ we define its Lipschitz norm

$$\|f\|_{\mathrm{Lip}}:=\|f\|_{\mathrm{Lip},\Lambda}:=\|f\|_{\mathrm{Lip},E}:=\sup_{\lambda\in\Lambda}\|f\|_E+|f|_{\mathrm{lip}}\,,$$
$$|f|_{\mathrm{lip}}:=|f|_{\mathrm{lip},\Lambda}:=|f|_{\mathrm{lip},E}:=\sup_{\lambda_1,\lambda_2\in\Lambda,\lambda_1\neq\lambda_2}\frac{\|f(\lambda_2)-f(\lambda_1)\|_E}{|\lambda_2-\lambda_1|}\,. \tag{1.3.4}$$

If a function $f\colon\widetilde{\Lambda}\subset\Lambda\to E$ is defined only on a subset $\widetilde{\Lambda}$ of Λ we shall still denote by $\|f\|_{\mathrm{Lip}}:=\|f\|_{\mathrm{Lip},\widetilde{\Lambda}}:=\|f\|_{\mathrm{Lip},E}$ the norm in (1.3.4), where the sup-norm and the Lipschitz seminorm are intended in $\widetilde{\Lambda}$, without specifying explicitly the domain $\widetilde{\Lambda}$.

If the Banach space E is the Sobolev space $\mathcal{H}^s$ then we denote more simply $\|\ \|_{\mathrm{Lip},\mathcal{H}^s}=\|\ \|_{\mathrm{Lip},s}$. If $E=\mathbb{R}$ then $\|\ \|_{\mathrm{Lip},\mathbb{R}}=\|\ \|_{\mathrm{Lip}}$.

If $A(\lambda)$ is a function, operator, …, that depends on a parameter λ, we shall use the following notation for the partial quotient

$$\frac{\Delta A}{\Delta\lambda}:=\frac{A(\lambda_2)-A(\lambda_1)}{\lambda_2-\lambda_1},\quad\forall\lambda_1\neq\lambda_2\,. \tag{1.3.5}$$

Given a family of functions, or linear self-adjoint operators $A(\lambda)$ on a Hilbert space H, defined for all $\lambda\in\widetilde{\Lambda}$, we shall use the notation

$$\mathfrak{d}_\lambda A(\lambda)\geq\beta\mathrm{Id}\quad\Longleftrightarrow\quad\frac{\Delta A}{\Delta\lambda}\geq\beta\mathrm{Id},\quad\forall\lambda_1,\lambda_2\in\widetilde{\Lambda},\lambda_1\neq\lambda_2\,, \tag{1.3.6}$$

where, for a self-adjoint operator,

$$A \geq \beta \mathrm{Id}$$

means as usual

$$(Aw, w)_H \geq \beta \|w\|_H^2, \quad \forall w \in H\,.$$

Given linear operators A, B we denote their commutator by

$$\mathrm{Ad}_A B := [A, B] := AB - BA\,. \tag{1.3.7}$$

We define $D_V := \sqrt{-\Delta + V(x)}$ and $D_m := \sqrt{-\Delta + m}$ for some $m > 0$.

- Given $x \in \mathbb{R}$ we denote by $\lceil x \rceil$ the smallest integer greater than or equal to x, and by $[x]$ the integer part of x, i.e., the greatest integer smaller than or equal to x;
- $\mathbb{N} = \{0, 1, \ldots\}$ denote the natural numbers, and $\mathbb{N}_+ = \{1, \ldots\}$ the positive integers;
- Given $L \in \mathbb{N}$, we denote by $[\![0, L]\!]$ the integers in the interval $[0, L]$;
- We use the notation $a \lesssim_s b$ to mean $a \leq C(s)b$ for some positive constant $C(s)$, and $a \sim_s b$ means that $C_1(s)b \leq a \leq C_2(s)b$ for positive constants $C_1(s), C_2(s)$;
- Given functions $a, b\colon (0, \varepsilon_0) \to \mathbb{R}$ we write

$$a(\varepsilon) \ll b(\varepsilon) \quad \Longleftrightarrow \quad \lim_{\varepsilon \to 0} \frac{a(\varepsilon)}{b(\varepsilon)} = 0\,. \tag{1.3.8}$$

In the Monograph we denote by $\mathbb{S}, \mathbb{F}, \mathbb{G}, \mathbb{M}$ subsets of the natural numbers $\mathbb{N}$, with

$$\mathbb{N} = \mathbb{S} \cup \mathbb{S}^c, \quad \mathbb{F} \cup \mathbb{G} = \mathbb{S}^c, \quad \mathbb{F} \cap \mathbb{G} = \emptyset, \quad \mathbb{F} \subset \mathbb{M}\,.$$

We refer to Chapter 4 for the detailed notation of operators, matrices, decay norms, etc.

For simplicity of notation we may write either $\xi \in \mathbb{R}^{\mathbb{S}}$ or $\xi \in \mathbb{R}^{|\mathbb{S}|}$.

Throughout the monograph we shall use the letter j to denote a space index: it may be

- j in $\mathbb{N}$, when we use the L^2 basis $\{\Psi_j\}_{j \in \mathbb{N}}$ defined in (1.1.5);
- $j = (j_1, \ldots, j_d)$ in $\mathbb{Z}^d$, when we use the exponential basis $\{e^{\mathrm{i} j \cdot x}\}_{j \in \mathbb{Z}^d}$.