Introduction

The main purpose of this series of lectures is to get acquainted with hyperbolic structures on surfaces and the classification of mapping classes. Soon enough, it will transpire that it is all about understanding curves on surfaces. The basic object of study is a surface Σ, always smooth, orientable, compact and connected. We define $\mathcal{S}(\Sigma)$ to be the set of non-contractible (smoothly) embedded circles in Σ, up to isotopy. This set is countable, since the fundamental group of Σ is. We refer to elements of $\mathcal{S}(\Sigma)$ as simple closed curves. Every diffeomorphism of Σ acts on $\mathcal{S}(\Sigma)$. This action is invariant under continuous deformations of a diffeomorphism. Therefore, the mapping class group, defined as the set of isotopy classes of orientation-preserving diffeomorphisms,

$$\mathrm{MCG}(\Sigma) = \mathrm{Diff}^+(\Sigma)/\mathrm{Diff}_0(\Sigma),$$

acts on the set $\mathcal{S}(\Sigma)$, as well. As we will see, the isotopy type of a diffeomorphism is determined by its action on simple closed curves.

Fact 1. *For a closed surface Σ of genus at least three,*

$$\mathrm{MCG}(\Sigma) < \mathrm{Perm}(\mathcal{S}(\Sigma)).$$

Surfaces of negative Euler characteristic carry hyperbolic structures. We will study the space $\mathcal{T}(\Sigma)$ of hyperbolic structures on a surface Σ, up to isometries isotopic to the identity map. A hyperbolic metric attributes a unique length to each isotopy class of closed curves, in turn defines an element of $\mathbb{R}_{>0}^{\mathcal{S}(\Sigma)}$.

Fact 2. *The natural map $l\colon \mathcal{T}(\Sigma) \to \mathbb{R}_{>0}^{\mathcal{S}(\Sigma)}$ is a proper embedding.*

The composition of the map $l\colon \mathcal{T}(\Sigma) \to \mathbb{R}_{>0}^{\mathcal{S}(\Sigma)}$ with the natural projection to the real projective space

$$\pi\colon \mathbb{R}_{>0}^{\mathcal{S}(\Sigma)} \to P(\mathbb{R}^{\mathcal{S}(\Sigma)})$$

is still an embedding, but not a proper one. The limit points, whose union we denote by $\mathcal{PMF}(\Sigma)$, correspond to a certain kind of degenerate length functions called (projective) measured foliations on the surface Σ. The homeomorphism types of the spaces $\mathcal{T}(\Sigma)$ and its compactification in $P(\mathbb{R}^{\mathcal{S}(\Sigma)})$ are

$$\mathcal{T}(\Sigma) \simeq \mathbb{R}^{6g-6}$$

and

$$\mathcal{T}(\Sigma) \cup \mathcal{PMF}(\Sigma) \simeq D^{6g-6},$$

where $g \geq 2$ denotes the genus of Σ and D^{6g-6} is a closed ball. This leads us to the third important statement.

Fact 3. *The mapping class group* $\mathrm{MCG}(\Sigma)$ *acts on $\mathcal{T}(\Sigma) \cup \mathcal{PMF}(\Sigma)$ by homeomorphisms. In particular, by Brouwer's fixed point theorem, each (class of) diffeomorphism $f\colon \Sigma \to \Sigma$ has a fixed point in $\mathcal{T}(\Sigma) \cup \mathcal{PMF}(\Sigma)$.*

There are two fundamentally different possibilities:

(1) f has a fixed point in $\mathcal{T}(\Sigma)$, in which case it is periodic, up to isotopy.

(2) f has a fixed point in $\mathcal{PMF}(\Sigma)$. Then f is either reducible or of pseudo-Anosov type, two features we will shortly explain.

It is the latter type which is the most interesting, and also generic, for that matter. Pseudo-Anosov maps $f : \Sigma \to \Sigma$ come with a real stretch factor $\lambda(f) > 1$. We will put a special emphasis on studying pseudo-Anosov maps with small stretch factors.

On our way, we will come across numerous beautiful results, old and new. Here is a somewhat random sample of theorems we will prove.

Theorem (Dehn [14]). *The mapping class groups* $\mathrm{MCG}(\Sigma)$ *are generated by Dehn twists along simple closed curves.*

Theorem (Basmajian [4]). *The length of the boundary curve of a hyperbolic surface* Σ *with one geodesic boundary component* $\partial\Sigma$ *is determined by its ortho-spectrum, in the following way:*

$$l(\partial\Sigma) = \sum_{\gamma \perp \partial\Sigma} 2\,\mathrm{arcsinh}\left(\frac{1}{\sinh(l(\gamma))}\right).$$

A similar identity involving only simple closed curves was later obtained by McShane [43]. This paved the way to Mirzakhani's famous counting result for the number of simple closed geodesics on a fixed hyperbolic surface [46]. The next result by Lanier and Margalit implies that pseudo-Anosov maps with a small stretch factor normally generate the mapping class group.

Theorem (Lanier and Margalit [35]). *Let* $N < \mathrm{MCG}(\Sigma)$ *be a non-trivial normal subgroup and let* $f \in N$ *be a pseudo-Anosov map with stretch factor* $\lambda(f)$*. Then* $\lambda(f) \geq \sqrt{2}$.

The structure of this manuscript is derived from the graduate course I gave at ETH in the autumn 2018. There are twelve chapters, each corresponding to a two hours lecture. Of the many literature sources that inspired these notes, three stand out: *Braids, links, and mapping class groups* by Birman [7], the *Primer on mapping class groups* by Farb and Margalit [19], and *Travaux de Thurston sur les surfaces* by Fathi, Laudenbach and Poénaru [20], of which there exists an English translation by Kim and Margalit [21]. I would also like to highlight the excellent books by Benedetti and Petronio [5], Casson and Bleiler [11], Ivanov [31], as well as Thurston's original work [60, 61].

The reader only interested in hyperbolic surfaces and mapping classes can stop reading in the middle, that is at the end of Chapter 6, and possibly add Chapters 10 and 12. Cutting it further down, I would recommend the hyperbolic part, Chapters 1–4, or the Dehn twist part, Chapters 5–6. For the most minimalistic approach, just read Chapter 2.