Chapter 1

Introduction

Orthogonal polynomials were introduced in the 18th century when Adrien M. Legendre studied the problem of gravitational attraction between a body and a sphere in his paper entitled "Sur l'attraction des sphèroïdes." Legendre proved the following statement: if the force of attraction exerted by a solid of revolution is known on an exterior point along its axis of revolution, then the force of attraction is also known for every point on the exterior of the solid. Here Legendre introduced a family of orthogonal polynomials $(P_n(x))_{n\geq 0}$ and he showed that the zeros of $P_n(x)$ are all real, simple, and located in the closed interval $[-1, 1]$ (see [2]). These polynomials can be represented by the Rodrigues formula

$$P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} \left(x^2 - 1\right)^n,$$

given by Olinde Rodrigues.

The Hermite polynomials $(H_n(x))_{n\geq 0}$ made their appearance between 1799 and 1825. Even though they are named in honor of Charles Hermite (1822–1901), it seems like the first person to consider them was Pierre-Simon Laplace who used them for the first time in his celebrated "Traité de mécanique céleste" to treat problems of the theory of probabilities. These polynomials were also studied by P. L. Chebyshev and finally by Hermite, who studied them extensively. The Hermite polynomials satisfy the following orthogonality relation:

$$\int_{-\infty}^{+\infty} H_n(x) H_m(x) e^{-x^2}\, dx = 2^n n! \sqrt{\pi} \delta_{n,m}.$$

Another well-known family of orthogonal polynomials, named after Edmond Nicolas Laguerre, are the Laguerre polynomials $(L_n^{(\alpha)}(x))_{n\geq 0}$ which satisfy the following orthogonality condition:

$$\int_0^{\infty} L_m^{(\alpha)}(x) L_n^{(\alpha)}(x) x^{\alpha} e^{-x}\, dx = \frac{\Gamma(n+\alpha+1)}{n!} \delta_{n,m}, \quad \alpha > -1.$$

These polynomials were first studied by Niels Henrik Abel and Joseph-Louis Lagrange, but it was Chebyshev who first dealt with them in more detail. In 1879, Laguerre used the particular case $\alpha = 0$ to study the integral $\int_x^{\infty} e^{-t} t^{-1} dt$ and found that these polynomials are solutions of the differential equation

$$xy'' + (x+1)y' = ny, \quad n \geq 0.$$

In some texts, the polynomials $(L_n^{(\alpha)}(x))_{n\geq 0}$ are also known as the Laguerre–Sonin polynomials, after Nikolai Yakovlevich Sonin who continued Sojotkin's work for $\alpha > -1$, discovering properties for those polynomials.

The German mathematician Karl Jacobi was the first to introduce a family of orthogonal polynomials without trying to solve a specific physical or mathematical problem. Jacobi introduced these polynomials in terms of hypergeometric functions, which had already been studied by a famous mathematician Carl F. Gauss. Jacobi defined them as

$$P_n^{(\alpha,\beta)}(x) = \frac{\Gamma(n+\alpha+1)}{\Gamma(\alpha+1)n!}\,{}_2F_1\left(-n, n+\alpha+\beta+1; \alpha+1, \frac{1-x}{2}\right),$$

with $\alpha > -1$ and $\beta > -1$. These polynomials are orthogonal with respect to the measure $d\mu(x) = (1-x)^\alpha(1+x)^\beta \chi_{[-1,1]}(x)dx$ where $\chi_{[-1,1]}$ is the characteristic function defined on the interval $[-1, 1]$.

The Laguerre, Hermite, and Jacobi polynomials are known as the families of classical orthogonal polynomials (see Chapter 9).

From the emergence of the classical orthogonal polynomials until today, the theory of orthogonal polynomials has grown exponentially mainly due to its numerous applications in physics, approximation theory, differential and difference equations, mechanics, and statistics (among others).

The above paragraphs have only been a small overview of the whole history behind orthogonal polynomials, so we invite the reader interested in deepening his/her knowledge on this subject to see, for example, [2, 4, 30, 37, 41, 52, 55] and the references therein.