Introduction

The motion of flows of fluids containing fine-grained solid inclusions is of great interest in a wide range of technical applications, ecology, and medicine. Such flows occur both in nature (for example, transfer of fine solid particles in rivers and seas, sandstorms, and dust storms) and in practical human activities (hydro and pneumatic transport, dust collectors, technological processes associated with the production of medicines, etc.). The motion of such mixtures of fluid with fine solid particles (we will call them suspensions) can be described using systems of equations of classical mechanics: the Navier–Stokes equations for the viscous fluid phase and the equations of solid dynamics for all particles with the condition of sticking a fluid to the surfaces of solid particles. This is in some sense a microscopic, mathematical model of suspension motion. However, this model is unsuitable in a practical sense for either numerical calculations or qualitative analysis of suspension behavior, due to the large number of particles, the smallness of their sizes, and the distances between them. Therefore, the development of a macroscopic model is required that can describe the motion of the mixture under consideration as a general continuous medium.

The traditional phenomenological approach of continuum mechanics, based on conservation laws for the mixture, does not lead to a closed system of equations for an associated effective continuous medium.

More suitable is the so-called microstructural approach, which consists in combining the phenomenological approach with a detailed study of fluid flow in the neighborhoods of particles, finding the local energy of fluid in these neighborhoods, and subsequent spatial homogenization ([1, 16]). This approach goes back to A. Einstein, who obtained, for the first time, a formula for the effective viscosity of a critically diluted suspension of solid spherical particles in a viscous incompressible fluid (see [48]). The approach was further developed in subsequent works in hydromechanics ([6, 10–13, 27–29, 32–34, 52, 58, 61, 65, 104, 107]). The main achievements were related to diluted (i.e., weakly concentrated) suspensions, where the sizes of particles are much less than the distances between nearest particles. On the other hand, there are almost no results for concentrated mixtures that describe their motion. This, according to the renowned specialist in fluid mechanics J. Batchelor, presents a true challenge to researchers ([12]).

Concerning the available works proposing various mathematical models of suspension motion with applications to the calculation of specific flows, we notice that these models were obtained at a physical level of rigor, often without sufficient justification and indication of areas of applicability. Moreover, these models often contradict each other. Therefore, the problem of constructing macroscopic models of motion of mixtures of fluids and fine solid particles is still very real now. A natural approach to this problem, which corresponds to a certain mathematical level of rigor, is to study the asymptotic behavior of suspension and to describe the possible asymptotic mode of its motion as the diameters of the particles and the distances between nearest particles tend to zero. Such modes can be described using homogenized (averaged) differential equations, which can then be considered as macroscopic models of suspension motion. This approach is very typical in the homogenization theory of partial differential equations – a branch of mathematics that has been intensively developed over the last 50 years ([15, 37, 66, 93–95, 106, 110]).

This monograph is devoted to the rigorous mathematical analysis of problems that arise in the study of microscopic models of the motion of suspensions from the standpoint of homogenization theory. The original model of motion is a system of equations consisting of the Navier–Stokes equations for a viscous incompressible fluid and the equations of rigid body dynamics, describing the collision-free motion of solid particles in a fluid with a stick boundary condition. Our main goal is to derive closed homogenized equations, describing the asymptotic behavior of solutions of this system when the diameters of the particles and the distances between neighboring particles tend to zero.

The problems of the homogenization of the Navier–Stokes equations have been considered in many mathematical works ([3–5, 23, 42, 51, 63, 67, 86, 87, 92, 98, 111, 116]). In particular, the asymptotic behavior of the solution of the Dirichlet problem, which describes the flow of a viscous incompressible fluid through a fixed porous medium, has been studied in detail. Homogenized equations obtained in this way model the Darcy or Brinkman laws for fluid flow in porous media, depending on the degree of porosity of the medium ([5, 30, 31, 62]). The homogenization of the Navier–Stokes equations in random perforated domains was described in [14, 43–46, 56]. However, it is impossible to directly apply these results to the homogenization of the original microscopic model of suspension dynamics since the positions of the particles, and consequently the perforated domain occupied by the fluid, change over time and are not known in advance. To cope with this problem, we use a natural approach consisting in dividing the problem into two simpler parts. This technique can be called a method of fixing particles: it is assumed that at any moment in time, the positions of the particles are known, and a special boundary-value problem for the stationary part of the Navier–Stokes equations (the model problem) is to be considered in the complementary domain.

It turns out that two qualitatively different asymptotic modes of suspension motion can be realized, depending on the parameters of the suspension: frozen particles mode and filtering particles mode. They correspond to two different types of the model problem. As the first part of the method of fixing particles, the asymptotic behavior of the solution of the corresponding model problem is studied. As a result, the homogenized system is derived, which actually describes the perturbation of the fluid by particles. Note that this system is not closed because it includes some unknown characteristics of the set of perturbing particles, namely the density of the mesoscopic viscosity tensor and the homogenized particle density for the frozen particles mode, and the distribution function of particles over coordinates, velocities, and sizes for the filtering particles mode.

The problem of the closure of this system in the case of concentrated suspensions remains open. This has been done only for weakly concentrated suspensions, for which an asymptotic expression for the mesoscopic viscosity tensor was obtained in terms of the distribution function of particles over coordinates, sizes, and orientations, and formulas for the forces and moments of forces acting on the particles from the fluid were found. Taking this into account, the closure is carried out using kinetic equations of Liouville type (or Fokker–Planck) for the distribution functions of particles ([54, 117]). It is clear that this natural method of deriving closed homogenized equations of suspension motion cannot be considered to be completely justified from the mathematical point of view. Consequently, closed systems of equations obtained by this method require an analysis of their consistency, i.e., compatibility and solvability. Such an analysis is partially done in the present monograph.

The monograph consists of eight chapters. In Chapter 1 we consider the original system of equations describing the motion of a suspension of solid small bodies (particles) in a viscous incompressible fluid. We introduce the notion of generalized solutions of initial–boundary-value problems for this system and discuss the question of existence of such solutions. A priori estimates for the solutions are derived, with the help of which the asymptotic behavior of the suspension is studied as the particle diameters and mean distances between nearest neighbors tend to zero. It is established that, depending on the relationship between these parameters, two qualitatively different modes of suspension motion can be realized: frozen particles mode, when the velocities of the particles coincide (asymptotically) with the mean velocity of the surrounding fluid, and filtering particles mode, when their velocities differ significantly from the mean velocity of the fluid. Two model problems, A and B, corresponding to these modes of motion are given. These problems are used in subsequent chapters for the derivation of closed homogenized equations, which describe the asymptotic modes and thus can be viewed as macroscopic models of suspension motion. We also introduce model problem C, which corresponds to the flow of a suspension of polarized (or magnetized) particles in frozen particles mode, in very strong electric (or magnetic) fields.

Chapter 2 deals with concentrated and weakly concentrated suspensions such that the mean distance between nearest neighboring particles is $O(\varepsilon)$, whereas the particle diameters are $O(\varepsilon^{\alpha})$, with $1 \leq \alpha < 3$. In this case, suspension motion corresponds to frozen particles mode. To derive the homogenized equations describing this mode, we have to study the asymptotic behavior of the solution of model problem A. For this we introduce the notion of a mesoscopic viscosity tensor, which allows us to formulate the main result (Theorems 2.1.1 and 2.2.1). We obtain uniform with respect to ε estimates of derivatives of solutions of the original problem, which, together with

Theorems 2.1.1 and 2.2.1, allows us to derive the main homogenized equation (still nonclosed), describing the perturbation of the velocity vector of the fluid in frozen particles mode.

Chapter 3 is devoted to the calculation of the limiting density of the mesoscopic viscosity tensor, which is involved in the homogenized equations in frozen particles mode. We present the viscosity tensor of a concentrated suspension in terms of the solution of cell problems (Theorem 3.1.1). An asymptotic formula is also obtained for the mean value of the mesoscopic tensor of weakly concentrated suspension ($1 < \alpha < 3$) of axisymmetric particles with random distribution of their diameters and orientations. In the case of spherical particles, the well-known Einstein formula for the viscosity of diluted suspensions follows from this asymptotic formula.

In Chapter 4 we present formulas for the forces and moments of forces acting on an axisymmetric particle from the flow around it and derive the equations of motion of its center of mass and the evolution of the orientation vector. We introduce the notion of the mean vector of particle orientation and derive an equation for its evolution. Taking into account the results of Chapters 2 and 3, this allows us to write out a closed system of equations of motion of a suspension of axisymmetric particles in frozen particles mode. In particular, in the case of spherical particles this system has the form

$$\rho\Big[\frac{\partial u}{\partial t} + (u\cdot\nabla)u\Big] - \sum_{i=1}^{3}\frac{\partial}{\partial x_i}\Big[\mu(\rho)\frac{\partial u}{\partial x_i}\Big] + \nabla p = f;$$

$$\operatorname{div} u = 0, \qquad \frac{\partial \rho}{\partial t} + (u\cdot\nabla)\rho = 0,$$

which is a generalization of the well-known model of an inhomogeneous fluid ([9, 72]). The existence of global weak solutions of initial–boundary-value problems for such systems is proved.

Chapter 5 deals with concentrated ($\alpha = 1$) and weakly concentrated ($1 < \alpha < 3$) suspensions of polarized (magnetized) particles in an external electric (magnetic) field. When the suspension is in motion, the moments of forces act on the particles from both the fluid and the external field, and taking account of these moments leads to more complicated homogenized equations, compared with those obtained in Chapters 2–4. The derivation of these equations is based on the study of the asymptotic behavior (as $\varepsilon \to 0$) of the solution of model problem C (Theorem 5.1.1). Of particular interest is the case when the particles are strongly elongated (flattened) and are affected by a strong external field forcing them, being in motion, to be strictly oriented along the field (perpendicular to the field). Then the homogenized equations of motion of the suspension acquire a qualitatively new form: they represent a nonstandard hydrodynamic model in which the stress tensor depends not only on the strain tensor (as in classical continuum mechanics), but also on the fluid vorticity tensor (Theorem 5.3.1).

In Chapter 6, critically diluted suspensions are considered, when the diameters of solid particles are of order ε^{α}, where ε is the mean distance between neighboring particles and $\alpha = 3$. This is a critical relation between sizes of particles and distances between them, at which particles can move with velocities different from the velocity of the surrounding fluid, but significantly disturbing its motion. For smaller particles ($\alpha > 3$), the perturbation is weak and vanishes as $\varepsilon \to 0$. If the specific density of the solid phase is much greater than the density of the fluid, then under this condition ($\alpha = 3$) the suspension moves in filtering particles mode. The derivation of the homogenized equations in this case is based on the study of the asymptotic behavior of the solution of model problem B (Theorem 6.1.1). As a result, we obtain a homogenized equation depending on the distribution function of particles over coordinates, velocities, and sizes. Complementing this equation with the kinetic Liouville equation for the distribution function, we arrive at a closed system of equations describing the motion of the suspension in filtering particles mode. If the particles are affected by the Brownian motion of molecules of the carrier fluid, then the equation closure is performed using the Fokker–Planck equation. In the case of spherical particles, the closed system has the form

$$\begin{aligned}
&\frac{\partial u}{\partial t} + (u \cdot \nabla)u - \nu \Delta u + \beta \int_0^1 \int_{\mathbb{R}^3} r(u - v) f \, dv \, dr - \nabla p = g; \\
&\operatorname{div} u = 0; \\
&\frac{\partial f}{\partial t} + (v \cdot \nabla) f + \operatorname{div}_v [\Gamma_r(u, v) f] = \sigma_r \Delta_v f, \quad 0 < r < 1; \\
&\Gamma_r(u, v) = \gamma r^{-2}(u - v) + g_1,
\end{aligned}$$

where $u(x,t)$ is the velocity of the carrier fluid; $f(x,v,r,t)$ is the distribution function of particles with coordinates x, velocities v, and reduced radii r; ν is the kinematic viscosity of the carrier fluid; $\sigma_r = \sigma r^{-5}$; β, γ, σ are constants; and

$$(u \cdot \nabla) = \sum_{i=1}^{3} u_i \frac{\partial}{\partial x_i}, \quad \Delta = \sum_{i=1}^{3} \frac{\partial^2}{\partial x_i^2}, \quad \Delta_v = \sum_{i=1}^{3} \frac{\partial^2}{\partial v_i^2}.$$

The solvability of such systems is studied in Theorems 6.5.1 and 6.5.2.

The homogenization methods developed in Chapters 2–5, and based on the notion of a mesoscopic viscosity tensor, are then applied in Chapters 7 and 8 to the study of small oscillations of mixtures of a viscous incompressible fluid and fine solid particles interacting through elastic forces. Such mixtures are the simplest models of complex fluids with a microstructure, both occurring in nature and having an artificial origin. These are, for example, colloidal suspensions, rubber, and polymer fluids ([24, 26, 77–79, 108]). Such complex media have a specific microstructure, a characteristic feature of which is the presence of the primary fluid with small inclusions distributed in it in the form of atoms, molecules, or small particles of an alien substance. The inclusions interact both with the main fluid and directly with each other via forces of different natures (electrostatic, elastic, van der Waals), thus forming flexible elastic grids, chains, etc. It is practically impossible to investigate the properties of such media within the framework of the original microscopic models due to their strong micro-inhomogeneity. Therefore, a primary problem is to construct macroscopic (homogenized) models that would adequately describe the properties of such media.

In Chapter 7, a homogenized system of equations is derived describing small oscillations of the mixture of a Newtonian fluid with solid particles whose diameters are $O(\varepsilon^\alpha)$ $(1 \leq \alpha < 3)$ and the interaction forces between them are $O(\varepsilon)$, where ε is the mean distance between nearest neighboring particles. We prove that in this case the asymptotic (as $\varepsilon \to 0$) oscillations of the mixture are of the frozen particles mode, and the homogenized system of equations has the form

$$\rho(x)\frac{\partial^2 u}{\partial t^2} - \sum_{n,p,q,r=1}^{3} \frac{\partial}{\partial x_p}\Big\{a_{npqr}(x)e_{qr}\Big[\frac{\partial u}{\partial t}\Big]\Big\}e^n$$
$$- \sum_{n,p,q,r=1}^{3} \frac{\partial}{\partial x_p}\Big\{b_{npqr}(x)e_{qr}[u]\Big\}e^n$$
$$- \int_{-\infty}^{t} \sum_{n,p,q,r=1}^{3} C_{npqr}(x,\tau)e_{qr}\Big[\frac{\partial}{\partial \tau}u(x,t-\tau)\Big]e^n \, d\tau - \nabla p = 0,$$
$$\operatorname{div} u = 0,$$

where $u(x,t)$ is the displacement vector of the mixture, $\{e^1, e^2, e^3\}$ is the orthonormal basis in $\mathbb{R}^3$, $e_{qr}[u] = \frac{1}{2}\big[\frac{\partial u_q}{\partial x_r} + \frac{\partial u_r}{\partial x_q}\big]$ is the strain tensor, and $\rho(x)$ is the limiting mass density of the mixture. The tensors $\{a_{npqr}\}$, $\{b_{npqr}\}$, $\{C_{npqr}\}$ are expressed in terms of the limiting density of the mesoscopic mixture tensor. This system is a typical representative of incompressible viscoelastic medium models (see, for example, [22, 78]).

In Chapter 8 we consider a mixture in which the sizes of solid particles have a critical order, $O(\varepsilon^3)$, and the distances between neighboring particles and the elastic forces of interaction between them are $O(\varepsilon)$. The asymptotic mode of oscillations of such a mixture corresponds to the filtering particles mode. In this case, the homogenized system of equations turns out to be a two-phase model (Theorem 8.2.1). For spherical particles, it has the form

$$\rho\frac{\partial v}{\partial t} - \mu\Delta v + C(x)(v - w) = \nabla p, \quad \operatorname{div} v = 0;$$
$$C(x)(w - v) - \sum_{n,p,q,r=1}^{3} \frac{\partial}{\partial x_p}\Big(a_{npqr}(x)e_{qr}\Big[\int_0^t w \, d\tau\Big]\Big)e^q = 0,$$

where $v(x,t)$ is the velocity of the fluid, $w(x,t)$ is the velocity of the solid phase, $C(x) = 6\pi\mu r(x)$, $r(x)$ is the limiting density of distribution of the radii of the particles, and $\{a_{npqr}(x)\}$ is the limiting density of the mesoscopic elastic tensor corresponding to the equilibrium state of the solid phase of the mixture.

This system can be interpreted as a generalization of the Brinkman law in the case of the flow of a viscous incompressible fluid through an array of particles of critical size elastically interacting with each other.

Summing up, it should be noted that in this book we attempt to rigorously derive macroscopic models of the motion of complex fluids with a microstructure of suspension type, using the methods of homogenization theory. It must be said that the problem of homogenization of the original system of equations, which is a microscopic model of the motion of a fluid with small solid particles, turned out to be rather complicated (and, perhaps, poorly posed). Indeed, the solvability of the original system is proved only for a small time interval $(0, T_\varepsilon)$, which decreases with decreasing parameter ε. Consequently, the initial object of homogenization seems to vanish in the limit as $\varepsilon \to 0$. For this reason, we make the following a priori assumption in the book: the solution $u_\varepsilon(x,t)$ of the initial–boundary-value problem of the original system of equations for any $\varepsilon > 0$ exists on a fixed time interval $[0, T]$, independent of $\varepsilon > 0$, and the distances between nearest particles at any moment of time $t \in [0, T]$ have the same order of smallness $O(\varepsilon)$ as at the initial moment. Under these conditions, the asymptotic behavior of $u_\varepsilon(x,t)$ as $\varepsilon \to 0$ is studied in order to derive the homogenized equations, which are a macroscopic model of suspension motion.

Another difficulty arising in the homogenization of such a microscopic model is that, due to particle motions, the region $\Omega_\varepsilon(t)$ occupied by the fluid phase changes in time and, in advance, is unknown, in contrast to the standard homogenization theory for elliptic and parabolic equations. Therefore, the homogenized problem posed in the book is divided into two subproblems. In the first, it is assumed that at any time t the domain $\Omega_\varepsilon(t)$ is known, and in it we consider the boundary-value problem for the stationary linear part of the Navier–Stokes equation, with certain boundary conditions on the surfaces of particles depending on the parameters of the mixture (model boundary-value problems A, B, and C). We study the asymptotic behavior of model problems as $\varepsilon \to 0$. In this case, we use the methods of homogenization that were developed in the monograph [95] for domains of arbitrary form, which are domains $\Omega_\varepsilon(t)$ (i.e., not periodic). As a result, we derive the homogenized equations that describe the perturbation of the fluid phase of the suspension by particles (see Sections 2.1, 2.4, 5.3, 6.1–6.3). But the obtained equations are not closed, and in order to close them it is necessary to derive the homogenized equation of motion of the solid phase of the suspension also. This is the second subproblem of homogenization of the initial equations of suspension motion. In the book, this problem has been solved only for dilute suspensions, when the particle diameters are much less than the distances between nearest neighbors. We obtain formulas for forces and moments of forces acting on a particle, moving in a linear fluid flow (see Sections 4.1, 4.2). Taking these formulas into account, we derive the evolution equation for the mean orientation vector of axisymmetric particles (Section 4.3) moving in frozen particles mode, and the homogenized motion of an ensemble of spherical particles in filtration mode is described using kinetic equations of Liouville type. Note that in the book the reasoning in the derivation of homogenized equations of motion of the solid phase of the suspension is carried out not quite rigorously from a mathematical point of view, but it corresponds to the level of rigor adopted in theoretical hydrodynamics.

The combination of homogenized equations of the solid and fluid phases leads to a closed system of equations, which is a macroscopic model of the dynamics of suspensions (see Sections 4.4, 5.3, 6.4). In Sections 4.5 and 6.5 the questions of compatibility and solvability of the obtained closed systems are studied.

Thus, the results concerning the homogenization of the fluid phase of the suspension have been obtained and presented in this book at a rigorous mathematical level (these are Chapters 1, 2, 3, 5, 6, 7, 8 and Section 4.5). The results concerning the homogenization of the solid phase correspond to the level of rigor adopted in theoretical hydrodynamics (these are Sections 4.1–4.4 and 6.4).

Most of the results presented in this book were published in [17–22, 80–90].

I am grateful to Maria Goncharenko for preparing the text for publication and translating the book into English.

March 2021 Evgen Ya. Khruslov