Preface

The present notes originated from a series of lectures given at ETH Zürich in the framework of a Nachdiplomvorlesung. Compactness and stability for nonlinear elliptic equations in the inhomogeneous context of closed Riemannian manifolds are investigated. We describe blow-up phenomena and present the progress made in the field over the past years. Special attention is devoted to the nonlinear stationary Schrödinger equation and to its critical formulation.

Over the whole book, the general background space is that of a smooth compact Riemannian n-manifold (M,g) without boundary. We refer to such manifolds as closed Riemannian n-manifolds. The generic equation we use as a model case is the stationary nonlinear Schrödinger equation but with this idea, that its potential has to be regarded as a varying object. In other words we look for the problem

$$\begin{aligned} &\Delta_g u + hu = u^{p-1} \ , \\ &\text{where } h \text{ is varying in } \mathcal{C} \ , \end{aligned} \tag{1}$$

and we aim to get uniform results with respect to $h \in \mathcal{C}$, where $\mathcal{C}$ is a given class of functions, e.g. like $\mathcal{C} = L^\infty, C^1$ or $C^{0,\theta}$ for some $\theta \in (0,1)$. In the above, $\Delta_g = -\mathrm{div}_g\nabla$ is the Laplace-Beltrami operator, and $p \in (2, 2^\star]$ is a pure power nonlinearity, where $2^\star = \frac{2n}{n-2}$ is the critical Sobolev exponent (assuming here that $n \geq 3$). The plus sign in front of the nonlinearity indicates that the nonlinear term competes with the Laplacian term. This is the difficult case in such equations, an elliptic analogue of the focusing case for dispersive models.

Let H^1 be the Sobolev space of functions in L^2 with one derivative in L^2. By the Sobolev embedding theorem, H^1 embeds continuously in L^p for all $p \leq 2^\star$. By the Rellich-Kondrakov theorem, these embeddings are compact when $p < 2^\star$, meaning that bounded sequences in H^1 have a subsequence which converges in L^p for $p < 2^\star$. On the other hand, by scale invariance, the embedding $H^1 \subset L^{2^\star}$ is never compact. As is well known, this implies that there is a serious difference between the subcritical world for which $p < 2^\star$, and the critical world for which $p = 2^\star$.

The rough question we use as a connecting thread through the book is:

how much is an equation like (1) *robust with respect to h ?*

The question has a very simple answer, though non-trivial, in the subcritical case. Compactness and stability hold true without any assumptions when $p < 2^\star$. This ceases to be the case in the critical setting, and we face there a fascinating landscape to which most of this book is devoted. The naive question we asked above then splits into the question of developing blow-up theories for equations like (1), and the questions of the compactness and stability of equations like (1). By blow-up theory we mean a theory that describes the blow-up behavior of blowing-up sequences of solutions of equations like (1). This will be a very general theory depending on the space in which we want the description to hold. By compactness and stability we mean results which either state that there are no blowing-up sequences for a given equation or, geometrically speaking, that solutions of perturbations of a given equation are close to solutions of the original unperturbed equation, or which, on the other hand, establish the existence of blowing-up sequences of solutions. This will very much depend on the geometry and the equation we consider.

The book is organized as follows. In Chapter 1, we discuss model equations related to the generic stationary nonlinear Schrödinger equation that we use as a model in the book, with this idea that its potential is a varying object. Chapter 2 is concerned with basic variational methods for solving nonlinear elliptic PDEs of the type we consider and the regularity issue. We discuss in Chapter 3 the L^p and H^1-theories for blow-up. These theories describe in L^p-spaces and H^1-spaces the asymptotic behavior of arbitrary blowing-up sequences of solutions associated with our equations. This is a priori analysis. Chapter 4 is concerned with describing several results on the opposite side of constructive analysis, where blowing-up sequences of solutions for the type of equations we consider are shown to exist. Different notions of stability for elliptic PDEs, including analytic stability and bounded stability, are discussed in Chapter 5. Chapter 6 is concerned with bounded stability. The C^0-theory for blow-up is described in Chapter 7. Chapter 8 is concerned with analytic stability and the notion of range of influence of blow-up points.

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