Introduction

The first stochastic process that has been extensively studied is the Brownian motion, named in honor of the botanist Robert Brown, who observed and described in 1828 the random movement of particles suspended in a liquid or gas. One of the first mathematical studies of this process goes back to the mathematician Louis Bachelier, in 1900, who presented in his thesis [2] a stochastic modelling of the stock and option markets. But, mainly due to the lack of rigorous foundations of probability theory at that time, the seminal work of Bachelier has been ignored for a long time by mathematicians. However, in his 1905 paper, Albert Einstein brought this stochastic process to the attention of physicists by presenting it as a way to indirectly confirm the existence of atoms and molecules. The rigorous mathematical study of stochastic processes really began with the mathematician Andrei Kolmogorov. His monograph [46] published in Russian in 1933 built up probability theory in a rigorous way from fundamental axioms in a way comparable to Euclid's treatment of geometry. From this axiomatic, Kolmogorov gives a precise definition of stochastic processes. His point of view stresses the fact that a stochastic process is nothing else but a random variable valued in a space of functions (or a space of curves). For instance, if an economist reads a financial newspaper because he is interested in the prices of barrel of oil for last year, then he will focus on the curve of these prices. According to Kolmogorov's point of view, saying that these prices form a stochastic process is then equivalent to saying that the curve that is seen is the realization of a random variable defined on a suitable probability space. This point of view is mathematically quite deep and provides existence results for stochastic processes as well as pathwise regularity results.

Joseph Doob writes in the introduction to his famous book "Stochastic processes" [19]:

A stochastic process is any process running along in time and controlled by probability laws…more precisely any family of random variables where a random variable … is simply a measurable function …

Doob's point of view, which is consistent with Kolmogorov's and built on the work by Paul Lévy, is nowadays commonly given as a definition of a stochastic process. Relying on this point of view that emphasizes the role of time, Doob's work, developed during the 1940s and the 1950s has quickly become one of the most powerful tools available to study stochastic processes.

Let us now describe the seminal considerations of Bachelier. Let X_t denote the price at time t of a given asset on a financial market (Bachelier considered a given quantity of wheat). We will assume that $X_0 = 0$ (otherwise, we work with $X_t - X_0$). The first observation is that the price X_t can not be predicted with absolute certainty.

It seems therefore reasonable to assume that X_t is a random variable defined on some probability space. One of the initial problems of Bachelier was to understand the distribution of prices at given times, that is, the distribution of the random variable $(X_{t_1}, \dots, X_{t_n})$, where $t_1, \dots, t_n$ are fixed.

The two following fundamental observations of Bachelier were based on empirical observations:

- If τ is very small then, in absolute value, the price variation $X_{t+\tau} - X_t$ is of order $\sigma\sqrt{\tau}$, where σ is a positive parameter (nowadays called the volatility of the asset).
- The expectation of a speculator is always zero[1] (nowadays, a generalization of this principle is called the absence of arbitrage).

Next, Bachelier assumes that for every $t > 0$, X_t has a density with respect to the Lebesgue measure, let us say $p(t, x)$. It means that if $[x - \varepsilon, x + \varepsilon]$ is a small interval around x, then

$$\mathbb{P}(X_t \in [x - \varepsilon, x + \varepsilon]) \simeq 2\varepsilon p(t, x).$$

The two above observations imply that for τ small,

$$p(t + \tau, x) \simeq \frac{1}{2} p(t, x - \sigma\sqrt{\tau}) + \frac{1}{2} p(t, x + \sigma\sqrt{\tau}).$$

Indeed, due to the first observation, if the price is x at time $t + \tau$, it means that at time t the price was equal to $x - \sigma\sqrt{\tau}$ or to $x + \sigma\sqrt{\tau}$. According to the second observation, each of these cases occurs with probability $\frac{1}{2}$.

Now Bachelier assumes that $p(t, x)$ is regular enough and uses the following approximations coming from a Taylor expansion:

$$\begin{aligned}
p(t + \tau, x) &\simeq p(t, x) + \tau \frac{\partial p}{\partial t}(t, x), \\
p(t, x - \sigma\sqrt{\tau}) &\simeq p(t, x) - \sigma\sqrt{\tau} \frac{\partial p}{\partial x}(t, x) + \frac{1}{2}\sigma^2 \tau \frac{\partial^2 p}{\partial x^2}(t, x), \\
p(t, x + \sigma\sqrt{\tau}) &\simeq p(t, x) + \sigma\sqrt{\tau} \frac{\partial p}{\partial x}(t, x) + \frac{1}{2}\sigma^2 \tau \frac{\partial^2 p}{\partial x^2}(t, x).
\end{aligned}$$

This yields the equation

$$\frac{\partial p}{\partial t} = \frac{1}{2}\sigma^2 \frac{\partial^2 p}{\partial x^2}(t, x).$$

This is the so-called heat equation, which is the primary example of a diffusion equation. Explicit solutions to this equation are obtained using the Fourier transform, and by using the fact that at time 0, p is the Dirac distribution at 0, it is

[1] Quoted and translated from the French: *It seems that the market, the aggregate of speculators, can believe in neither a market rise nor a market fall, since, for each quoted price, there are as many buyers as sellers.*

computed that

$$p(t,x) = \frac{e^{-\frac{x^2}{2\sigma^2 t}}}{\sigma\sqrt{2\pi t}}.$$

It means that X_t has a Gaussian distribution with mean 0 and variance σ^2. Let now $0 < t_1 < \cdots < t_n$ be fixed times and $I_1, \dots, I_n$ be fixed intervals. In order to compute $\mathbb{P}(X_{t_1} \in I_1, \dots, X_{t_n} \in I_n)$ the next step is to assume that the above analysis did not depend on the origin of time, or, more precisely, that the best information available at time t is given by the price X_t. This leads to the following computation:

$$\begin{aligned}
\mathbb{P}(X_{t_1} \in I_1, X_{t_2} \in I_2) &= \int_{I_1} \mathbb{P}(X_{t_2} \in I_2 | X_{t_1} = x_1) p(t_1, x_1) dx_1 \\
&= \int_{I_1} \mathbb{P}(X_{t_2 - t_1} + x_1 \in I_2 | X_{t_1} = x_1) p(t_1, x_1) dx_1 \\
&= \int_{I_1 \times I_2} p(t_2 - t_1, x_2 - x_1) p(t_1, x_1) dx_1 dx_2,
\end{aligned}$$

which is easily generalized to

$$\begin{aligned}
&\mathbb{P}(X_{t_1} \in I_1, \dots, X_{t_n} \in I_n) \\
&= \int_{I_1 \times \cdots \times I_n} p(t_n - t_{n-1}, x_n - x_{n-1}) \dots p(t_2 - t_1, x_2 - x_1) p(t_1, x_1) dx_1 dx_2 \dots dx_n.
\end{aligned} \tag{0.1}$$

In many regards, the previous computations were not rigorous but heuristic. One of our first motivations is to provide a rigorous construction of this object X_t on which Bachelier worked and which is called a Brownian motion.

From a rigorous point of view the question is: Does there exist a sequence of random variables $\{X_t, t \geq 0\}$ such that $t \to X_t$ is continuous and such that the property (0.1) is satisfied? Chapter 1 will give a positive answer to this question. We will see how to define and construct processes. In particular we will prove the existence of Brownian motion and then study several of its properties.

Chapter 1 sets the foundations. It deals with the basic definitions and results that are required to rigorously deal with stochastic processes. We introduce the relevant σ-fields and prove the fundamental Daniell–Kolmogorov theorem which may be seen as an infinite-dimensional version of the Carathéodory extension of measure theorem. It is the basic theorem to prove the existence of a stochastic process. However, despite its importance and usefulness, the Daniell–Kolmogorov result relies on the axiom of choice and as such is non-constructive and does not give any information or insight about the stochastic process that has been proved to exist. The

Kolmogorov continuity theorem fills one of these gaps and gives a useful criterion to ensure that we can work with a process whose sample paths are continuous. Chapter 1 also includes a thorough study of continuous martingales. We focus on Doob's theorems: The stopping theorem, the regularization result and the maximal inequalities. Martingale techniques are essential to study stochastic processes. They give the tools to handle stopping times which are naturally associated to processes and provide the inequalities which are the cornerstones of the theory of stochastic integration which is presented in Chapter 5.

Chapter 2 is devoted to the study of the most important stochastic process: The Brownian motion. As a consequence of the Daniell–Kolmogorov and Kolmogorov continuity theorems, we prove the existence of this process and then study some of its most fundamental properties. From many point of views, Brownian motion can be seen as the continuous random walk in continuous time. This is made precise at the end of the chapter where we give an alternative proof of the existence of the Brownian motion as a limit of suitably rescaled random walks.

Chapter 3 is devoted to the study of Markov processes. Our goal is to emphasize the role of the theory of semigroups when studying Markov processes. More precisely, we wish to understand how one can construct a Markov process from a semigroup and then see what are the properties inherited from the semigroup to the sample path properties of the process. We will particularly focus on the class of Feller–Dynkin Markov processes which are a class of Markov processes enjoying several nice properties, like the existence of regular versions and the strong Markov property. We finish the chapter with a study of the Lévy processes which form an important subset of the class of Feller–Dynkin Markov processes.

Chapter 4 can be thought an introduction to the theory of symmetric Dirichlet forms. As we will see, this theory and the set of tools attached to it belong much more to functional analysis than to probability theory. The basic problem is the construction of a Markov semigroup and of a Markov process only from the generator. More precisely, the question is: Given a diffusion operator L, does L generate a Markov semigroup $\boldsymbol{P}_t$ and if yes, is this semigroup the transition semigroup of a continuous Markov process? We will answer positively this question in quite a general framework under the basic assumption that L is elliptic and essentially self-adjoint with respect to some Borel measure.

Chapter 5 is about stochastic calculus and its applications to the study of Brownian motion. Stochastic calculus is an integral and differential calculus with respect to Brownian motion or more generally with respect to martingales. It allows one to give a meaningful sense to integrals with respect to Brownian motion paths and to define differential equations driven by such paths. It is not a straightforward extension of the usual calculus because Brownian motion paths are not regular enough, they are only γ-Hölder continuous with $\gamma < 1/2$. The range of applications of the stochastic calculus is huge and still growing today. We mention in particular that the applications to mathematical finance have drawn a lot of attention on this calculus. Actually, most of the modern pricing theories for derivatives on financial markets are based on Itô–Döblin's formula which is the chain rule for stochastic calculus.

Chapter 6 deals with the theory of stochastic differential equations. Stochastic differential equations are the differential equations that correspond to Itô's integration theory. They give a very powerful tool to construct a Markov process from its generator. We will prove the basic existence and uniqueness results for such equations and quickly turn to the basic properties of the solution. One of the problems we are mostly interested in is the existence of a smooth density for the solution of a stochastic differential equation. This problem gave birth to the so-called Malliavin calculus which is the study of the Sobolev regularity of Brownian functionals and that we study in some detail.

Chapter 7 is an introduction to the theory of rough paths that was developed by Lyons in the 1990s. The theory is deterministic. It allows one to give a sense to solutions of differential equations driven by very irregular paths. Stochastic differential equations driven by Brownian motions are then seen as a very special case of rough differential equations. The advantage of the theory is that it goes much beyond the scope of Itô calculus when dealing with differential systems driven by random noises and comes with powerful estimates.