Preface

Cluster algebras can now be regarded as a field of study in its own right, as is borne out by the allocation of a Mathematics Subject Classification number, 13F60, in the 2010 revision. But they did not exist before the current century: they were introduced in the seminal article [67] of S. Fomin and A. Zelevinsky which appeared in print in the Journal of the American Mathematical Society in 2002. So what has contributed to this phenomenal growth?

The notion of a cluster algebra captures a number of interrelated ideas in one beautiful setting. In one sense, a cluster algebra can be thought of as some kind of discrete dynamical system. It is defined using combinatorial data which is then mutated arbitrarily to produce altered copies of the original data. But it is also a Lie-theoretic object: the cluster algebras of finite type have a classification by the Dynkin diagrams, in a similar way to many other objects, including simple Lie algebras over the complex numbers and finite crystallographic reflection groups. The structure of the latter runs right through the corresponding cluster algebras.

A cluster algebra is also a representation-theoretic object: attempts to categorify cluster algebras, i.e. to model them using associated categories, have led to new developments in the representation theory of algebras. A cluster algebra is also a combinatorial object: the combinatorial fascination begins with interesting formulas for numbers of clusters and frieze patterns and goes on into combinatorial geometry, with new insights into generalized associahedra. In fact, a cluster algebra is also a geometric object, with strong connections to the the theory of Riemann surfaces.

In recent years, a number of excellent survey articles explaining how cluster algebras are representation theoretic objects have been published (more details are given in the introduction), but there is scope for a more detailed description of some of their other aspects. This book arose out of a desire to explain some of these. The large size of this growing field means that it is not possible to include everything that is being done, but the aim is to give a good idea of a number of interesting developments in the field.

It is intended that these notes should be accessible to graduate students. Where proofs are not included, detailed references are given to allow those who wish to learn more to delve more deeply into the subject. Thus the aim of these notes is to give an introduction to cluster algebras in their own right and to some of the many aspects of cluster algebras.

In Chapter 1, we introduce cluster algebras and describe the motivation for them. Chapter 2 gives key definitions via matrices and mutation, and first properties of cluster algebras, leading onto the definition via exchange patterns in Chapter 3, where a description of the relationship between the approaches via matrices and polynomials is given. As a preparation for the finite type classification theory, Chapter 4 is a short introduction to reflection groups. Chapter 5 gives a description of cluster algebras of finite type and their classification. Chapter 6 is an introduction to the generalized associahedra, which are polytopes associated to cluster algebras of finite type. Chapter 7 covers the notion of periodicity in cluster algebras, in the context of the Laurent phenomenon and integer sequences, as well as the categorical periodicity in Keller's proof of the periodicity conjecture of Zamolodchikov. Chapter 8 looks at quivers of finite mutation type, including cluster algebras arising from marked Riemann surfaces, and Chapter 9 considers the cluster algebra structure of the homogeneous coordinate ring of a Grassmannian.

These lecture notes began as a Nachdiplom (graduate) lecture course on Cluster Algebras given at the Department of Mathematics at the Eidgenössische Technische Hochschule (ETH) Zürich (the Swiss Federal Institute of Technology) in Zürich in Spring 2011, while I was a guest of the Forschungsinstitut für Mathematik (FIM, Institute for Mathematical Research). I am very grateful for the welcome I received in the Department and the Institute. I would also like to thank everyone who attended and contributed to the course and made it such an enjoyable experience for me.

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This book is dedicated to the memory of Andrei Zelevinsky, who changed a significant part of the mathematical landscape.

Robert Marsh, University of Leeds, August 2013.