1 Introduction

1.1 Motivation for cluster algebras

Cluster algebras are commutative algebras, defined combinatorially by an iterated mutation process. They were introduced by S. Fomin and A. Zelevinsky [67] mainly with the long-term aim of modelling the multiplicative structure of the dual canonical basis associated to the quantized enveloping algebra of a symmetrizable Kac-Moody Lie algebra $\mathfrak{g}$. The fundamental problem they were working on (which is still open), was an explicit description of this basis.

The quantized enveloping algebra $U_q(\mathfrak{g})$ was introduced by V. G. Drinfel'd [44] and M. Jimbo [101], and can be regarded as the q-analogue of the corresponding universal enveloping algebra of $\mathfrak{g}$. G. Lusztig [116, 117, 118] introduced the canonical basis of the quantized enveloping algebra $U_q(\mathfrak{n})$ of the positive part, $\mathfrak{n}$, of $\mathfrak{g}$, in the case where $\mathfrak{g}$ is a symmetric Kac-Moody algebra, and Kashiwara independently introduced the global crystal basis of $U_q(\mathfrak{n})$ in the symmetrizable case. The bases were shown to coincide in the symmetric case in [87], and have beautiful properties, including positivity properties, good representation-theoretic properties and an intriguing combinatorial structure. However, it is very difficult to describe them explicitly. Studying this problem has led to a lot of interesting mathematics.

A complete description of the canonical basis has been given in only a few cases. The bases in types A_1 and A_2 appear already in G. Lusztig's first paper on the canonical basis [116]. Type A_3 is described in [119, 162] and type B_2 in [163]. In type A_4, partial information is known [32, 94, 95, 123].

The canonical basis induces a basis in $\mathbb{C}_q[N]$, the quantum deformation of the algebra $\mathbb{C}[N]$ of regular functions on the pro-unipotent group N associated to $\mathfrak{n}$, which is known as the dual canonical basis, and this basis appears to be more tractable than the canonical basis itself. If N_- denotes the opposite pro-unipotent group to N, set $N(w) = N \cap (w^{-1}N_-w)$ for each element w of the corresponding Weyl group. Then, in [80] it is shown that the coordinate ring $\mathbb{C}[N(w)]$ is a cluster algebra. Furthermore, it is shown that the dual of Lusztig's semicanonical basis (see [121] for the definition) induces bases in each $\mathbb{C}[N(w)]$ containing all cluster monomials: in other words, the cluster structure of $\mathbb{C}[N(w)]$ is relevant to the semicanonical basis, which has a number of key properties in common with the canonical basis. In fact, the semicanonical basis and canonical basis coincide in types A_n for $1 \leq n \leq 4$ [79]. Furthermore, it is shown in [81] that the quantum coordinate ring $\mathbb{C}_q[N(w)]$ is a quantum cluster algebra in the sense of [25] (note that quantum cluster structures have also been studied in a Poisson geometric context in [57, 58, 59]). For another source of information on the dual canonical basis in this context, see [108] and references therein.

Roughly speaking, the idea is to understand the dual canonical basis (of $\mathbb{C}_q[N(w)]$) via its maximal q-commuting subsets. We should start with such a subset, and then try to find more subsets by a process of mutation in which elements are exchanged for new ones. If the process works well, a combinatorial object (such as a quiver, or matrix), can be associated to each q-commuting subset, governing the way in which the mutation process takes place, as well as being mutated itself each time mutation takes place. The set of generators for each q-commuting subset is known as a cluster, and mutation replaces one such generator with a new one. See [81] and references therein for more details concerning this particular quantum cluster algebra structure (see also [22, §6]). There has also been a lot of recent work on bases in cluster algebras per se, e.g. [34, 42, 47, 133, 142].

A second key motivation for the definition of cluster algebras was the notion of total positivity. A real matrix is totally positive if all of its minors are positive, including, for example, all of the entries. Such matrices were studied in a mechanical context in [78]. For an introduction to the relationship between total positivity and cluster algebras, see [61], where the example of G/N, with $G = SL_n(\mathbb{C})$ and N a maximal unipotent subgroup, is discussed in some detail. The coordinate ring $\mathbb{C}[G/N]$ is a cluster algebra. A matrix representing an element in G/N is totally positive if all of its flag minors are positive (i.e., all minors whose column set is an initial set of columns). It is not necessary to check all flag minors are positive in order to check total positivity; instead it is enough to check $(n-1)(n+2)/2$ minors. Categorizing such sets of minors (i.e. total positivity criteria) leads again to the notion of a cluster algebra: such sets behave well under a notion of mutation similar to that described above for the dual canonical basis (and, in fact, this is not a coincidence; for a discussion of the relationship between the canonical basis and total positivity see [120]).

A third key motivation for cluster algebras was the behaviour of certain sequences defined by rational functions which have integer entries. Often such sequences have the property that the nth term is a Laurent polynomial in the initial terms, although a priori it is only a rational function. The integer property then follows, in the case where the initial terms are all set equal to 1. Particularly nice examples of this are the Somos-n sequences (see e.g. [77]) for small n. From this point of view, a cluster algebra should be considered as a kind of recurrence on a regular tree (a tree all of whose vertices have the same valency) in which the formula for the recurrence depends on the vertex of the tree in a well-defined way. The Laurent phenomenon for cluster algebras [67, Thm. 3.1] states that any cluster variable (obtained by arbitrary iterations of the recurrence) is a Laurent polynomial in the initial variables. For more information on the Laurent phenomenon (and, in particular, the Somos sequences from this perspective), see [68]. See also [75] and the recent developments [112, 113].

As well as the fields described above, cluster algebras have been applied to a number of other areas. Cluster categories were introduced in [23] and, independently in type A, in [27], and give a certain kind of categorification of a cluster algebra. A generalization was given in [2]. Cluster categories have had applications to the representation theory of finite dimensional algebras and, in particular, to tilting theory. A more quiver representation-theoretic approach, using the notion of a quiver with potential, is available in [40]. There has been a lot of work in this field, and for more information on this we refer to the surveys [3, 22, 104, 105, 115, 146].

We also refer to two useful computer software packages for cluster algebras, B. Keller's Java applet for quiver mutation [106] and the Sage package on cluster algebras and quivers by G. Musiker and C. Stump [134].

There are a number of relationships between cluster algebras and other areas, and we give references to some of the articles: ad-nilpotent ideals in Borel subalgebras of simple Lie algebras [141], combinatorial geometry (associahedra) [35], discrete integrable systems [75, 103, 136] (and many more), Donaldson-Thomas invariants [107, 109, 110, 135], frieze patterns [5, 7, 8, 14, 15, 19, 88], hyperbolic 3-manifolds [138], BPS quivers and quantum field theories [1], Riemann surfaces [52, 65], Seiberg duality [18, 129, 161] (see also [75, §11]), RNA secondary structure combinatorics [126], shallow water waves and the KP equation [111], Teichmüller theory and Poisson geometry [33, 55, 57, 58, 59, 83, 99], and tropicalization [137]. Space prevents making a list of all references here: there are many more interesting articles on the subject. The list here is meant to give a flavour of the subject, and is not intended to be exhaustive. However, apologies are made for any omissions that should have been included. The *Cluster algebras portal* [60] is a good source of information.

1.2 Some recurrences

Let $k \geq 1$ be an integer, and consider the recurrence defined by $f_1 = x$, $f_2 = y$ and

$$f_{n+1} = \begin{cases} \frac{f_n+1}{f_{n-1}} & \text{if } n \text{ is odd;} \\ \frac{f_n^k+1}{f_{n-1}} & \text{if } n \text{ is even .} \end{cases}$$

If $k = 1$, we obtain the sequence

$$x, y, \frac{y+1}{x}, \frac{x+y+1}{xy}, \frac{x+1}{y}, x, y, \ldots .$$

For $k = 2, 3$, we obtain the sequences:

$$x, y, \frac{y^2+1}{x}, \frac{x+y^2+1}{xy}, \frac{(x+1)^2+y^2}{xy^2}, \frac{x+1}{y}, x, y, \ldots$$

and

$$x, y, \frac{y^3+1}{x}, \frac{x+y^3+1}{xy}, \frac{(x+1)^3+y^3(y^3+2+3x)}{x^2y^3}, \frac{(x+1)^2+y^3}{xy^2},$$
$$\frac{(x+1)^3+y^3}{xy^3}, \frac{x+1}{y}, x, y, \ldots$$

Firstly, by definition, the entries in the sequences are rational functions in x and y, but they turn out to have the stronger property of being Laurent polynomials. Secondly, we obtain sequences of period 5, 6 and 8, for $k = 1, 2$ and 3 respectively.

Furthermore, the denominators appearing seem to be following a kind of pattern (reminiscent of root systems of rank 2). These properties can be explained by the theory of cluster algebras introduced by S. Fomin and A. Zelevinsky [67]; see Example 2.1.7. These examples turn out to correspond to cluster algebras of finite type: see Chapter 5.

The recurrence with $k = 1$ is known as the *pentagon recurrence*, and can be attributed to R. C. Lyness or N. H. Abel (see [64, §1.1] for more details).

1.3 Somos recurrences

Choose an integer $r \geq 2$, and consider the recurrence:

$$x_n x_{n+r} = \begin{cases} x_{n+1}x_{n+r-1} + x_{n+2}x_{n+r-2} + \dots x_{n+\frac{r}{2}} x_{n-\frac{r}{2}} & \text{if } r \text{ is even;} \\ x_{n+1}x_{n+r-1} + x_{n+2}x_{n+r-2} + \dots x_{n+\frac{r-1}{2}} x_{n+\frac{r+1}{2}} & \text{if } r \text{ is odd,} \end{cases}$$

with initial conditions $x_1 = \cdots = x_r = 1$. This is known as the *Somos-r recurrence*. For example, if $r = 4$, the recurrence is defined by

$$x_n x_{n+3} = x_{n+1}x_{n+3} + x_{n+2}^2.$$

The first terms of the sequence are:

$$1, 1, 1, 1, 2, 3, 7, 23, 59, 314, 1529, 8209, \dots.$$

It is immediate from the definition that the Somos-r sequence consists of rational numbers. If $r = 4, 5, 6$ or 7 then it is known that the entries are all integers. For $r = 4$ or 5, this was proved independently by a number of authors in 1990, including J. L. Malouf [122, §1], Enrico Bombieri, and Dean Hickerson, according to J. Propp [145]. See also [77]. Propp also mentions that the cases $r = 6$ and $r = 7$ were proved by Dean Hickerson and Ben Lotto respectively in 1990.

We can treat the first r terms of the Somos-r sequence as indeterminates. Then, it was shown by S. Fomin and A. Zelevinsky [68], for $r = 4, 5, 6, 7$, that the nth term of the Somos-r sequence can be written as a Laurent polynomial in the first r terms. The proof used the theory of cluster algebras. It follows from this that the entries in the sequence, with initial conditions as above, are integers, giving an independent proof of this integrality.

According to entry A030127 in [140], the Somos-r sequences for for $r \geq 8$ are not integral, e.g. $x_{17} \notin \mathbb{Z}$ for Somos-8. This entry is the list of first terms in Somos-r sequences which are non-integral for $r \geq 8$. There are also interesting connections with elliptic curves: see [26, 50].

1.4 Why study cluster algebras?

Here we list some reasons for studying cluster algebras.

- An interest in the representation theory of quivers and finite dimensional algebras: cluster algebras give new families of algebras and new ways to compare categories (cluster-tilting theory).
- An interest in root systems and Weyl groups. For example, cluster algebras of finite type correspond to crystallographic root systems, and have a Dynkin classification.
- An interest in discrete integrable systems. In a certain sense, cluster algebras can be considered as such systems and there are interesting integrals on related recurrences.
- An interest in combinatorics. There is some beautiful combinatorics associated to cluster algebras, e.g. via root systems or representation theory, including generalized associahedra. The type A associahedron arises from the associativity rule, and has vertices corresponding to different ways of bracketing a sum or product.
- An interest in Riemann surfaces and Teichmüller theory. Many cluster algebras have a combinatorial description in terms of the geometry of surfaces, giving new perspectives on the representation theory mentioned above.

1.5 Some notation

Throughout the book, we will use the following notation, for real numbers x and integers a, b with $a \leq b$.

$$
\begin{aligned}
[x]_+ &= \max(x, 0);\\
[x]_- &= \min(0, x);\\
\operatorname{sgn}(x) &= \begin{cases} 1 & \text{if } x > 0;\\ 0 & \text{if } x = 0;\\ -1 & \text{if } x < 0; \end{cases}\\
[a, b] &= \{a, a+1, \ldots, b\}.
\end{aligned}
$$