Chapter 1
Introduction

In [T13] we dealt with the Navier-Stokes equations

$$\partial_t u + (u, \nabla)u - \Delta u + \nabla P = 0 \qquad \text{in } \mathbb{R}^n \times (0, \infty), \tag{1.1}$$

$$\operatorname{div} u = 0 \qquad \text{in } \mathbb{R}^n \times (0, \infty), \tag{1.2}$$

$$u(\cdot, 0) = u_0 \qquad \text{in } \mathbb{R}^n, \tag{1.3}$$

where $u(x,t) = \big(u^1(x,t), \ldots, u^n(x,t)\big)$ is the unknown velocity and $P(x,t)$ the unknown (scalar) pressure, $2 \le n \in \mathbb{N}$. Recall $\partial_t = \partial/\partial t$, $\partial_j = \partial/\partial x_j$ if $j = 1, \ldots, n$, and that the vector-function $(u, \nabla)u$ has the components

$$\big[(u, \nabla)u\big]^k = \sum_{j=1}^n u^j \partial_j u^k, \qquad k = 1, \ldots, n, \tag{1.4}$$

whereas, as usual,

$$\operatorname{div} u = \sum_{j=1}^n \partial_j u^j, \qquad \nabla P = (\partial_1 P, \ldots, \partial_n P). \tag{1.5}$$

By (1.2) one has

$$(u, \nabla)u = \operatorname{div}(u \otimes u), \qquad \operatorname{div}(u \otimes u)^k = \sum_{j=1}^n \partial_j (u^j u^k). \tag{1.6}$$

This reduces (1.1)–(1.3), now in the strip $\mathbb{R}^n \times (0, T)$ with $T > 0$, to

$$\partial_t u - \Delta u + \mathbb{P} \operatorname{div}(u \otimes u) = 0 \qquad \text{in } \mathbb{R}^n \times (0, T), \tag{1.7}$$

$$u(\cdot, 0) = u_0 \qquad \text{in } \mathbb{R}^n. \tag{1.8}$$

Here $\mathbb{P}$ is the Leray projector,

$$(\mathbb{P} f)^k = f^k + R_k \sum_{j=1}^n R_j f^j, \qquad k = 1, \ldots, n, \tag{1.9}$$

based on the (scalar) Riesz transforms R_k,

$$R_k g(x) = i \Big(\frac{\xi_k}{|\xi|} \widehat{g} \Big)^\vee (x) = c_n \lim_{\varepsilon \downarrow 0} \int_{|y| \ge \varepsilon} \frac{y_k}{|y|^{n+1}} g(x-y) \, \mathrm{d}y, \quad x \in \mathbb{R}^n. \tag{1.10}$$

In (1.7), (1.8) there is no need to care about (1.2) any longer. But if in addition $\operatorname{div} u_0 = 0$ then $\operatorname{div} u = 0$ in our context (mild solutions based on fixed point arguments). This well-known reduction of (1.1)–(1.3) to (1.7), (1.8) may also be found in [T13, Section 6.1.3, pp. 196-198]. The vector equation (1.7), (1.8) can be reduced to the nonlinear scalar heat equation

$$\partial_t u(x,t) - D\, u^2(x,t) - \Delta u(x,t) = 0, \qquad x \in \mathbb{R}^n,\ 0 < t < T, \tag{1.11}$$

$$u(x,0) = u_0(x), \qquad x \in \mathbb{R}^n, \tag{1.12}$$

on the one hand and the mapping properties of R_j and $\mathbb{P}$ in the considered function spaces on the other hand. Here

$$D\, f = \sum_{j=1}^{n} \partial_j f. \tag{1.13}$$

We dealt with the Cauchy problem (1.11), (1.12) in the context of *local spaces* $\mathcal{L}^r A^s_{p,q}(\mathbb{R}^n)$, [T13, Theorem 5.24, p. 183], and of *global spaces* $A^s_{p,q}(\mathbb{R}^n)$, [T13, Theorem 5.36, p. 189], under the crucial assumption that the underlying spaces $\mathcal{L}^r A^s_{p,q}(\mathbb{R}^n)$ and $A^s_{p,q}(\mathbb{R}^n)$ are multiplication algebras. This is ensured if $s + r > 0$ for local spaces and $s > n/p$ (and some limiting spaces with $s = n/p$) for global spaces. The reduction of (1.7), (1.8) to (1.11), (1.12) requires in addition that the Riesz transforms R_j are linear and bounded maps in the underlying spaces. This applies to the global spaces

$$A^s_{p,q}(\mathbb{R}^n), \qquad 1 < p < \infty, \quad 0 < q \le \infty, \quad s \in \mathbb{R}, \tag{1.14}$$

[T13, Theorem 1.25, p. 17] where the additional restriction $1 < q < \infty$ for F-spaces mentioned there is not necessary (as a consequence of Theorem 3.52 below). Then one obtains satisfactory solutions for (1.7), (1.8) in the global spaces

$$A^s_{p,q}(\mathbb{R}^n), \qquad 1 < p < \infty, \quad 1 \le q \le \infty, \quad s > n/p, \tag{1.15}$$

(and some limiting cases with $s = n/p$). We refer the reader to [T13, Theorem 6.7, p. 203] (where $1 < q < \infty$ for F-spaces can be replaced by $1 \le q \le \infty$ as covered by Corollary 5.4 below). We could not find a counterpart in terms of the local spaces $\mathcal{L}^r A^s_{p,q}(\mathbb{R}^n)$ and replaced as a substitute the Leray projector $\mathbb{P}$ in (1.7) by the truncated Leray projector $\mathbb{P}_{\psi^2}$ based on the truncated Riesz transforms

$$R_{\psi,k} f = i\Big(\psi\, \frac{\xi_k}{|\xi|}\, \widehat{f}\Big)^\vee, \qquad k = 1, \ldots, n, \tag{1.16}$$

where

$$\psi \in C^\infty(\mathbb{R}^n), \quad \psi(x) = 0 \text{ if } |x| \le 1/2 \quad \text{and} \quad \psi(y) = 1 \text{ if } |y| \ge 1, \tag{1.17}$$

[T13, pp. 199/200, Theorem 6.10, p. 205]. Hence, one removes the infrared (or low frequency) part of solutions of (1.7), (1.8). This point has also been discussed in

[T13, p. 193, 199-201]. At that time we tried to find a way to deal with Navier-Stokes equations or with (1.7), (1.8) also in the context of the local spaces $\mathcal{L}^r A^s_{p,q}(\mathbb{R}^n)$. But it came out quite recently that the Riesz transform (1.10) cannot be extended from $D(\mathbb{R}^n)$ or $S(\mathbb{R}^n)$ to a linear and bounded operator acting in the local Morrey spaces $\mathcal{L}^r_p(\mathbb{R}^n) = \mathcal{L}^r L_p(\mathbb{R}^n)$, $1 < p < \infty$, $-n/p \le r < 0$, [RoT13, Theorem 1.1(i)]. We refer the reader also to Theorem 2.22 and Remark 2.23 below. On the one hand one can take this observation as a justification of the above truncation. But on the other hand one knows now that R_k are linear and bounded maps,

$$R_k : \quad \mathring{L}^r_p(\mathbb{R}^n) \hookrightarrow \mathring{L}^r_p(\mathbb{R}^n) \quad \text{and} \quad L^r_p(\mathbb{R}^n) \hookrightarrow L^r_p(\mathbb{R}^n), \tag{1.18}$$

$1 < p < \infty$, $-n/p \le r < 0$, in the global Morrey spaces $L^r_p(\mathbb{R}^n) = L^r L_p(\mathbb{R}^n)$ and in the completion of $S(\mathbb{R}^n)$ in $L^r_p(\mathbb{R}^n)$, denoted as $\mathring{L}^r_p(\mathbb{R}^n)$, [RoT13, Theorem 1.1], Theorem 2.22 and Remark 2.23 below. We refer the reader also to [RoT14]. It is crucial for us and the main motivation of this book that (1.18) can be extended to some *hybrid spaces* $L^r A^s_{p,q}(\mathbb{R}^n)$ (being smaller than the local spaces $\mathcal{L}^r A^s_{p,q}(\mathbb{R}^n)$). As far as properties are concerned these spaces are between *local* and *global* spaces. This may justify calling them *hybrid* spaces. In particular if

$$1 < p < \infty, \quad 0 < q \le \infty, \quad s \in \mathbb{R} \quad \text{and} \quad -n/p \le r < 0, \tag{1.19}$$

then one has by Theorem 3.52 below

$$R_k : \quad L^r A^s_{p,q}(\mathbb{R}^n) \hookrightarrow L^r A^s_{p,q}(\mathbb{R}^n), \qquad k = 1, \ldots, n, \tag{1.20}$$

whereas the local spaces $\mathcal{L}^r A^s_{p,q}(\mathbb{R}^n)$ do not have this property. In addition $L^r A^s_{p,q}(\mathbb{R}^n)$ are multiplication algebras if $s + r > 0$ (as their local counterparts $\mathcal{L}^r A^s_{p,q}(\mathbb{R}^n)$). Then one can extend a corresponding theory for the nonlinear heat equations (1.11), (1.12), now in terms of the hybrid spaces $L^r A^s_{p,q}(\mathbb{R}^n)$, to the Navier-Stokes equations. We tried to find in [T13] related assertions in the context of the local spaces $\mathcal{L}^r A^s_{p,q}(\mathbb{R}^n)$. Now it is clear that this is impossible, but it is also clear that one has a satisfactory theory with hybrid spaces $L^r A^s_{p,q}(\mathbb{R}^n)$ in place of the local spaces $\mathcal{L}^r A^s_{p,q}(\mathbb{R}^n)$. This extends corresponding assertions from $A^s_{p,q}(\mathbb{R}^n) = L^{-n/p} A^s_{p,q}(\mathbb{R}^n)$ to $L^r A^s_{p,q}(\mathbb{R}^n)$.

Chapter 2 deals mainly with local and global Morrey spaces $\mathcal{L}^r_p(\mathbb{R}^n)$, $\mathring{\mathcal{L}}^r_p(\mathbb{R}^n)$, $L^r_p(\mathbb{R}^n)$, $\mathring{L}^r_p(\mathbb{R}^n)$ and their (pre)duals. We follow closely [RoT13, RoT14] complemented by

$$L^r_p(\mathbb{R}^n) \hookrightarrow L_p(\mathbb{R}^n, \mu_\alpha), \quad \mu_\alpha = w_\alpha \mu_L, \quad 1 < p < \infty, \quad -n/p < r < 0, \tag{1.21}$$

where μ_L is the Lebesgue measure and $w_\alpha(x) = (1 + |x|^2)^{\alpha/2}$ with $-n < \alpha < -n - rp$ is a Muckenhoupt weight $w_\alpha \in \mathcal{A}_p(\mathbb{R}^n)$. Then $R_k g(x)$ according to (1.10) is well-defined for $x \in \mathbb{R}^n$ a.e., also in its integral version. Finally we characterize some of these spaces in terms of Haar wavelets. In Chapter 3 we introduce the hybrid spaces $L^r A^s_{p,q}(\mathbb{R}^n)$ and collect some basic properties needed later on. This can be done largely in the same way as in [T13] for the local spaces $\mathcal{L}^r A^s_{p,q}(\mathbb{R}^n)$ mostly without additional efforts. Only occasionally we add a further argument. We observe that

$$L^r A^s_{p,q}(\mathbb{R}^n) = A^{s,\tau}_{p,q}(\mathbb{R}^n) \qquad \text{with} \quad \tau = \frac{1}{p} + \frac{r}{n} \tag{1.22}$$

for all admitted parameters s, p, q and $-n/p \le r < \infty$. The spaces $A^{s,\tau}_{p,q}(\mathbb{R}^n)$ have been studied in great detail in the book [YSY10], the survey [Sic12] and the underlying papers. There one finds many other properties which will not be repeated here. One may also consult [T13, pp. 38/39, Section 2.7.3, pp. 101-107]. There is one crucial exception needed to prove (1.20). Then we rely on

$$\|f \,|L^r A^s_{p,q}(\mathbb{R}^n)\| \sim \|f \,|L^r \dot{A}^s_{p,q}(\mathbb{R}^n)\| + \|f \,|L^r_p(\mathbb{R}^n)\| \tag{1.23}$$

if

$$1 < p < \infty, \quad 0 < q \le \infty, \quad s > 0, \quad -n/p \le r < 0. \tag{1.24}$$

Here $L^r \dot{A}^s_{p,q}(\mathbb{R}^n)$ are homogeneous hybrid spaces (we do not need the homogeneous spaces themselves but only their homogeneous norms in the context of the inhomogeneous spaces $L^r A^s_{p,q}(\mathbb{R}^n)$). For these homogeneous spaces (or their norms) one has the Fourier multiplier assertion

$$\|(h\widehat{f})^\vee |L^r \dot{A}^s_{p,q}(\mathbb{R}^n)\| \le c \Big(\sup_{|\alpha|\le k, x\in\mathbb{R}^n} |x|^{|\alpha|} \big|D^\alpha h(x)\big| \Big) \|f \,|L^r \dot{A}^s_{p,q}(\mathbb{R}^n)\| \tag{1.25}$$

of Michlin type with $k \in \mathbb{N}$ sufficiently large (specified later on). This is essentially covered by [YaY10, Theorem 4.1, p. 3819]. We refer also to [YYZ12, Theorem 1.5, p. 6] and the recent survey [YaY13a]. This can be applied to R_k with $h = \xi_k/|\xi|$. Then (1.20) with (1.19) follows essentially from (1.23) and (1.18), (1.25). This may be considered as the basic observation of what follows. Afterwards we return in Chapter 4 to the nonlinear heat equations (1.11), (1.12) and transfer assertions available so far in the context of the local spaces $\mathcal{L}^r A^s_{p,q}(\mathbb{R}^n)$ to their hybrid counterparts $L^r A^s_{p,q}(\mathbb{R}^n)$ (again essentially without any additional efforts) complemented by some new observations. In Chapter 5 we deal with the Navier-Stokes equations (1.7), (1.8) in hybrid spaces $L^r A^s_{p,q}(\mathbb{R}^n)$ extending a corresponding theory in [T13] for the spaces $A^s_{p,q}(\mathbb{R}^n) = L^{-n/p} A^s_{p,q}(\mathbb{R}^n)$ to $L^r A^s_{p,q}(\mathbb{R}^n)$. This extension applies not only to the obtained assertions, but also to the underlying technicalities. In particular (1.23) is the Morreyfied version of

$$\|f \,|A^s_{p,q}(\mathbb{R}^n)\| \sim \|f \,|\dot{A}^s_{p,q}(\mathbb{R}^n)\| + \|f \,|L_p(\mathbb{R}^n)\| \tag{1.26}$$

if

$$0 < p < \infty, \quad 0 < q \le \infty, \quad s > \sigma_p = n\big(\tfrac{1}{p} - 1\big)_+, \tag{1.27}$$

[T92, Theorem 2.3.3, p. 98]. Furthermore, (1.25) with

$$\dot{A}^s_{p,q}(\mathbb{R}^n) = L^{-n/p} \dot{A}^s_{p,q}(\mathbb{R}^n) \tag{1.28}$$

is covered by [T83, Theorem 5.2.2, p. 241]. We refer the reader also to [T13, Theorem 1.25, p. 17]. The final Chapter 6 is to some extent independent of the main bulk of this book. It deals with Haar wavelets, Faber bases and sampling in the context of the hyperbolic cross and spaces with dominating mixed smoothness and their relations to solutions of Navier-Stokes equations, global in time, for large initial data.