

# 1 Introduction

The purpose of this lecture series is to describe the *representation variety* — or *character variety* — of the fundamental group  $\pi_1(S)$  of a closed connected surface  $S$  of genus greater than 2, with values in a Lie group  $G$ . This character variety is roughly defined to be

$$\text{Rep}(\pi_1(S), G) := \text{Hom}(\pi_1(S), G)/G,$$

where  $G$  acts on  $\text{Hom}(\pi_1(S), G)$  by conjugation.

These character varieties have been heavily studied in the context of gauge theories — mainly in the case of compact groups — and hyperbolic geometry, when  $G = \text{PSL}(2, \mathbb{R})$  or  $G = \text{PSL}(2, \mathbb{C})$ .

In these lectures we shall be mainly interested in the topology of these varieties and their symplectic structure, which was discovered by Atiyah, Bott, and Goldman, without touching upon their interpretation in the theory of Riemann surfaces.

In the preliminary chapters, we give a crash course on surfaces, vector bundles and connections. While there is no new material here, we insist on describing very early both differential geometric and combinatorial aspects of the objects that we are interested in.

These various points of view allow us to give several models of the character variety, generalising the familiar picture about first cohomology groups being described alternatively using de Rham, Čech, or simplicial cohomology or as the space of homomorphisms of the fundamental group into  $\mathbb{R}$ .

We then describe the smooth structure of these character varieties as well as their tangent spaces. To explain the volume form on these tangent spaces we need to, following Witten, introduce the Reidemeister torsion.

When  $G = \text{PSL}(2, \mathbb{R})$ , we prove the Milnor–Wood Inequality and discuss connected components of the character variety.

We finally introduce the symplectic structure on these character varieties and prove Witten’s formula, which computes their symplectic volume in the case of a compact groups. We introduce two important algebras of observables, the first one consisting of Wilson loops or the other of spin networks, and compute their Poisson bracket, thus introducing the Goldman algebra.

In the last chapter, we turn to the integrality of the symplectic form and the relation with 3-manifolds and the Chern–Simons invariant.

Each chapter closes with a section giving general references and further reading.

These notes correspond to lectures given at ETH Zürich and Orsay at a beginning graduate level. The students were supposed to have only elementary knowledge of differential geometry and topology. Accordingly, these notes are quite informal: in the first chapters, we do not insist on the details of the differential geometric constructions and refer to classical textbooks, while in the more advanced chapters we are sometimes constrained to giving proofs only in special cases, hoping that this

gives the flavor of the general proofs. A typical example is the study of the properness of the action by conjugation of  $G$  on  $\text{Hom}(\pi_1(S), G)$ : while the general case requires a certain knowledge of algebraic groups and their actions on algebraic varieties, we only treat here the case of  $G = \text{SL}(n, \mathbb{R})$  resorting to elementary arguments on matrices.

None of the results presented here are new, but the presentation and proofs sometimes differ from other expositions.

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