Chapter 1
Introduction to Spectral Geometry

From P.-S. Laplace to E. Beltrami

The *Laplace operator* was first introduced by P.-S. Laplace (1749–1827) for describing celestial mechanics (the notation Δ is due to G. Lamé). For example, in our three-dimensional (Euclidean) space the Laplace operator (or just *Laplacian*) is the linear differential operator:

$$\Delta\colon \begin{cases} \mathcal{C}^2(U) \longrightarrow \mathcal{C}^0(U) \\ f \longmapsto \Delta f := \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}, \end{cases}$$

where U is an open set of $\mathbb{R}^3$. This operator can be generalized to a Riemannian manifold (M, g): this generalization is called the *Laplace–Beltrami operator* and is denoted by the symbol Δ_g. The study of this operator and in particular the study of its spectrum is called *spectral geometry*. The Laplace–Beltrami operator is very useful in many fields of physics, in particular in all diffusion processes:

- Fluid mechanics: the *Navier–Stokes equations*

$$\begin{cases} \frac{\partial u}{\partial t} + (u.\nabla)u - \nu\Delta u = -\nabla p \\ \operatorname{div}(u) = 0; \end{cases}$$

- potential theory and gravity theory (with Newton potential): the *Laplace equation* and the *Poisson equation*

$$\Delta u = f;$$

- heat diffusion process: the *heat equation*

$$\frac{\partial u}{\partial t}(x,t) - \Delta u(x,t) = f(x,t);$$

- wave physics: the *wave equation*

$$\frac{\partial^2 u}{\partial t^2}(x,t) - \Delta u(x,t) = 0;$$

- quantum physics: the *Schrödinger equation*

$$i\hbar\frac{\partial\psi}{\partial t}(x,t) = -\Delta\psi(x,t) + V(x)\psi(x,t);$$

- etc. ...

Let us mention that this operator appears also in:

- computer science: in particular, in computer vision (blob detection)
- economy: financial models, *Black–Scholes equations*
- etc. ...

Moreover, spectral geometry is an inter-disciplinary field of mathematics; it involves

- analysis of ODE and PDE
- dynamical systems: classical and quantum completely integrable systems, quantum chaos, geodesic flows and Anosov flows on (negatively curved) manifolds (for example, Ruelle resonances are related to the spectrum of the Laplacian, see the recent articles [Fa-Ts1], [Fa-Ts2]).
- geometry and topology (the main purpose of these notes is to explain this)
- geometric flow: the parabolic behaviour of scalar curvature

$$\frac{\partial R_{g(t)}}{\partial t} = \Delta R_{g(t)} + 2\left|\mathrm{Ric}_{g(t)}\right|^2;$$

  the Ricci flow (see also Sections 7.8.3 and 7.8.4)

$$\frac{\partial g}{\partial t} = -2\mathrm{Ric}_{g(t)}$$

- probability: the Brownian motion on a Riemannian manifold (M,g) is defined to be a diffusion on M (generator operator is given by $\frac{1}{2}\Delta_g$).
- etc. ...

Meaning of the Laplacian operator

To get a better feeling about the Laplacian operator, consider for example a one-dimensional $\mathcal{C}^3$ function $u:\mathbb{R}\to\mathbb{R}$. The mean value of u on the compact set $[-h,h]$ is given by

$$\overline{u} := \frac{1}{2h}\int_{-h}^{h} u(x)\,dx.$$

Now, using the Taylor expansion of u around the origin we get: for all $x \in [-h, h]$,

$$u(x) = u(0) + u'(0)x + u''(0)\frac{x^2}{2} + u'''(0)\frac{x^3}{6} + o(x^4).$$

Therefore,

$$\overline{u} = \frac{1}{2h}\left(\int_{-h}^{h} u(0) + u'(0)x + u''(0)\frac{x^2}{2} + u'''(0)\frac{x^3}{6}\,dx\right) + o(h^4)$$

i.e., $\overline{u} = u(0) + \frac{u''(0)}{12}h^2 + o(h^4)$, hence $\overline{u} - u(0) = \frac{u''(0)}{12}h^2 + o(h^4)$. In other words,

$$\Delta u(0) = u''(0) = \frac{12}{h^2}\left(\overline{u} - u(0)\right) + o(h^2),$$

thus the Laplacian of u measures the difference between the function u at 0 and the mean value of u on the neighbourhood $[-h, h]$.

Another way to understand this interpretation is to use the *finite difference method*: for example, on the domain $I = [0, 1]$ and for an integer $N > 0$ consider the discretization grid of I given by $\{t_i := ih,\ i \in \{0 \ldots N\}\}$ with $h := \frac{1}{N}$. Thus using the Taylor expansion of u around the point t_i we have

$$u(t_{i+1}) = u(t_i + h) = u(t_i) + u'(t_i)h + u''(t_i)\frac{h^2}{2} + u'''(t_i)\frac{h^3}{6} + o(h^4)$$

and

$$u(t_{i-1}) = u(t_i - h) = u(t_i) - u'(t_i)h + u''(t_i)\frac{h^2}{2} - u'''(t_i)\frac{h^3}{6} + o(h^4),$$

whence

$$\Delta u(t_i) = u''(t_i) = \frac{u_{i-1} - 2u_i + u_{i+1}}{h^2} + o(h^2)$$

where $u_k := u(t_k)$. The principal term $u_{i-1} - 2u_i + u_{i+1}$ represents the difference between the value of the function u at the point t_i and the values of u on the grid-neighborhood of t_i. In particular, the discretization of the problem $-u'' = f$ on I with the boundary conditions $u(0) = u(1) = 0$ is the linear equation $AX = B$ with

$$A = \begin{pmatrix} 2 & -1 & & & 0 \\ -1 & 2 & \ddots & & \\ & -1 & \ddots & \ddots & \\ & & \ddots & 2 & -1 \\ 0 & & & -1 & 2 \end{pmatrix}$$

and

$$X = \begin{pmatrix} u_1 \\ \vdots \\ u_N \end{pmatrix}, \qquad B = \begin{pmatrix} f_1 \\ \vdots \\ f_N \end{pmatrix}.$$

More generally: *the Laplace–Beltrami operator at a point x measures the difference of the mean value of f on a neighbourhood of x and the value of the function at the point x.*

Main Topics in Spectral Geometry

Given a (compact) Riemannian manifold (M, g), we can associate to it the (linear) unbounded operator $-\Delta_g$. This operator is self-adjoint and its spectrum is discrete (see Chapter 4): namely, the spectrum consists of an increasing sequence $(\lambda_k(M))_k$ of real eigenvalues of finite multiplicity such that $\lambda_k(M)$ as $k \to +\infty$. We denote this spectrum by

$$\mathrm{Spec}(M, g) := (\lambda_k(M))_k .$$

In others words, for any integer $k \geq 0$ there exists a non trivial function u_k on M (eigenfunction) such that

$$-\Delta_g u_k = \lambda_k(M) u_k .$$

The spectral theory of the Laplacian on a compact Riemannian manifold (M, g) is in particular interested in the connection between the spectrum $\mathrm{Spec}(M, g)$ and the geometry of the manifold (M, g). Indeed there are many deep connections between spectrum and geometry. Therefore spectral geometry studies such connections. The main topics in spectral geometry can be split into two types.

Direct problems

The main questions in direct problems are (see Chapter 5):

Question. *Can we compute (exactly or not) the spectrum* $\mathrm{Spec}(M, g)$*? And (or): can we find properties on the spectrum* $\mathrm{Spec}(M, g)$*?*

Then the principle of direct problems is to compute or find some properties on the spectrum of a compact Riemannian manifold (M, g). Obviously the question to compute the spectrum of Laplacian (or for other operators) arises in a lot of problems: analysis of PDE, dynamical systems, mathematical physics, differential geometry, probability, etc. See, for example, Section 4.1.3 for a concrete application in quantum dynamics.

For example, the first non-null eigenvalue $\lambda_\star(M)$ plays a very important role in Riemannian geometry, and one of the main questions is to find a lower bound for $\lambda_\star(M)$ depending on the geometric properties of the manifold, e.g. the dimension n, the volume $\mathrm{Vol}(M,g)$, the curvature R, etc. (see Section 5.2.1):

$$\lambda_\star(M) \geq a(n, \mathrm{Vol}(M,g), R, \cdots).$$

Another example of (asymptotic) computation (based on the Weyl formula, see Section 7.6) is

$$\lambda_k(M) \underset{k\to+\infty}{\sim} \left(\frac{(2\pi)^n}{B_n \mathrm{Vol}(M,g)}\right)^{\frac{2}{n}} k^{\frac{2}{n}};$$

here n is the dimension of the compact manifold (M,g) and $B_n := \frac{\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2}+1)}$.

Inverse problems

The data of a metric on a Riemannian manifold determine completely the Laplacian Δ_g and therefore its spectrum $\mathrm{Spec}(M,g)$. Hence the spectrum is a *Riemannian invariant*: if two Riemannian manifolds (M,g) and (M',g') are isometric, then they are isospectral, i.e., $\mathrm{Spec}(M,g) = \mathrm{Spec}(M',g')$. Conversely the main question is

Question. *Does the data of the spectrum* $\mathrm{Spec}(M,g)$ *determine the "shape" of the manifold* (M,g)*?*

In other words: which geometric information can we deduce from the spectrum? For example, we have the classical heat invariants; indeed the spectrum of the manifold (M,g) determines

- the dimension of (M,g)
- the volume of (M,g)
- the integral of the scalar curvature Scal_g over (M,g).

Another main topic in inverse problems is

Question. *What sequences of real numbers can be spectra of a compact manifold?*

A simpler version of this question is: Let M be a fixed manifold, given a finite increasing sequence of real numbers $0 < a_1 \leq a_2 \leq \cdots \leq a_N$ does, there exists a metric g such that the k^{th} first eigenvalues of (M,g) are equal to $0 < a_1 \leq a_2 \leq \cdots \leq a_N$? Y. Colin de Verdière proved in 1987 that the answer is positive (see Section 7.4).

The next important example of inverse problem concerns the length spectrum (see Section 7.2). The length spectrum of a compact Riemannian manifold (M,g)

is the set of lengths of closed geodesics on (M, g) counted with multiplicities. As it turns out, the spectrum of the manifold determines the length spectrum. Spectral theory is also an important tool for understanding the relationships between the formalism of classical mechanics and that of quantum mechanics:

- The formalism of classical mechanics on a Riemannian manifold is expressed in terms of geodesics.
- The formalism of quantum mechanics on a Riemannian manifold is expressed in terms of the Laplace–Beltrami operator.

The question of isospectrality in Riemannian geometry may be traced back to H. Weyl in 1911–1912 and became popularized thanks to M. Kac's article of 1966 [Kac1]. The famous sentence of Kac "*Can one hear the shape of a drum?*" refers to this type of isospectral problem. The exact formulation of the isospectrality question is

Question. *If two Riemannian manifolds* (M, g) *and* (M', g') *are isospectral, are they isometric?*

The answer is negative and was given first by J. Milnor in 1964 (see Section 7.3). In 1984 and 1985, respectively C. Gordon, E.N. Wilson [Go-Wi] and T. Sunada [Sun] gave a systematic construction of counter-examples. In 1992, C. Gordon, D. Webb, and S. Wolpert [GWW1] gave the first planar counter-example.

The main classical references on spectral geometry are the book of M. Berger, P. Gauduchon and E. Mazet [BGM], the book of P. Bérard [Bér8][Bér], of I. Chavel [Cha1] and the book of S. Rosenberg [ROS].

Organization of the book

The present book is a basic introduction to spectral geometry. The reader is assumed to have a good grounding in functional analysis and differential calculus. Chapter 2 discusses the fundamental notions of spectral theory for compact and unbounded operators. Chapter 3 is a review of differentiable manifolds and Riemannian geometry, including the definition of Laplace–Beltrami operator. In Chapter 4 we define the spectrum of the Laplace–Beltrami operator on a Riemannian manifold and then we present the minimax principle and some geometrical consequences. Chapter 5 discusses principles underlying the treatment of direct problems of spectral geometry, including some exact computations of spectrum. In Chapter 6 we present a topological perturbation result for eigenvalues of a manifold. Chapter 7 is devoted to inverse problems in spectral geometry; in particular, at the end of the chapter we introduce briefly some results of conformal geometry in dimension 2 and 3.