Preface

This book originated from an unpublished, very preliminary manuscript of ours of 2001. Our plan was to use it as a basis for a comprehensive treatment of the defocusing nonlinear Schrödinger equation on the circle. Later, Jürgen Pöschel joined us in this project and a substantially revised, but in many respects still incomplete version of these notes has been available since 2009. Unfortunately, other commitments prevented Jürgen from further participating in the project and he decided to withdraw from authorship. We wish to express our appreciation and gratitude for all the contributions he has made and for generously allowing us to use them.

This monograph is concerned with the theory of integrable partial differential equations. It offers a concise case study of the *normal form theory* of such equations for the defocusing nonlinear Schrödinger equation on the circle – one of the most important nonlinear integrable PDEs, both in view of its applications, in particular to nonlinear optics, and of the fact that this equation comes up as an important model equation in more than one space dimension as well.

To be more specific, our starting point is the defocusing nonlinear Schrödinger equation on the circle – dNLS for short – considered as an *infinite-dimensional integrable system* admitting a complete set of independent integrals in involution. We show that dNLS admits a single, global, real-analytic system of coordinates – the cartesian version of action-angle coordinates, also referred to as Birkhoff coordinates, – such that the dNLS Hamiltonian becomes a function of the actions alone. In fact, these coordinates work simultaneously for all Hamiltonians in the dNLS hierarchy.

Similar results were obtained for the Korteweg-de Vries equation (KdV), another important integrable PDE – see [26]. The existence of *global* Birkhoff coordinates is a special feature of dNLS and KdV. However, for many integrable PDEs, *local* Birkhoff coordinates may be constructed in parts of phase space satisfying appropriate conditions by developing our approach further. Specifically, in [30] this has been shown for the *focusing* nonlinear Schrödinger equation, which due to the presence of features of hyperbolic dynamics in certain parts of phase space is known *not* to admit global Birkhoff coordinates.

The global coordinates make it evident that all solutions of the dNLS equation on the circle are periodic, quasi-periodic, or almost periodic in time. They also provide a convenient tool to handle small Hamiltonian perturbations far away from the equilibrium ([19], [24]) by extending the KAM theory in a suitable way. As the Birkhoff coordinates are global and real-analytic, it suffices to check the pertinent nondegeneracy conditions of the NLS frequencies $\omega_n, n \in \mathbb{Z}$, as functions of the actions near the equilibrium. This is achieved by computing the Birkhoff normal form of the dNLS Hamiltonian up to order four [33]. The situation differs from more conventional applications of KAM methods to integrable PDEs in that the frequencies ω_n and ω_{-n} satisfy $\omega_n - \omega_{-n} = O(1)$ as $n \to \infty$, leading to additional difficulties in obtaining a KAM theorem which would be valid for small but otherwise arbitrary Hamiltonian perturbations. We plan to include a concise treatment of such a KAM theorem in a future expanded version of this book, where we also want to present applications of the normal form theory to the study of qualitative features of solutions of dNLS.

The book is most closely related to the monograph "KdV & KAM" [26] where the normal form theory is developed for KdV and then applied to obtain a KAM theorem for Hamiltonian perturbations. Furthermore it is also closely related to important earlier work on the construction of actions and angles for dNLS [41], [42], which we have used as a basis for getting global Birkhoff coordinates. We also note that elements of normal form theory for integrable PDEs with a Lax pair formulation can be found in [32]. With the scope of proving a KAM theorem for Hamiltonian perturbations, Kuksin constructs coordinates near finite-dimensional invariant tori, based on the Its-Matveev formula.

This book is intended not only for the handful of specialists working at the intersection of integrable PDEs and Hamiltonian perturbation theory, but also for researchers farther away from these fields. In fact, with the aim of reaching out to graduate students as well, we have made the book self-contained. In particular, we present a detailed study of the spectral theory of self-adjoint Zakharov-Shabat operators on an interval which appear in the Lax pair formulation of dNLS, filling in this way a long standing gap in the literature. In addition, we included several appendices – some of them, we believe, of independent interest – on topics from complex analysis on Hilbert spaces, Hamiltonian formalism on infinite-dimensional phase spaces, infinite products, and some spectral results from functional analysis. Also, we wrote the book in a modular manner where each of its chapters as well as its appendices may be read independently of each other.

This book has taken many years to complete, and during this long time we have benefited from discussions and collaborations with many friends and colleagues. We would like to thank all of them, first and foremost Peter Topalov, with whom we have obtained, among others, results on the normal form theory for the *focusing* nonlinear Schrödinger equation. We would also like to thank our PhD students for their valuable feedback and help, in particular Hasan Inci, Jan Molnar and Yannick Widmer.

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Last but not least, we thank our families for all their encouragement throughout these years.

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