Overview

In this treatise we consider the *defocusing nonlinear Schrödinger equation* on the real line,

$$\mathrm{i}\partial_t u = -\partial_x^2 u + 2\,|u|^2\,u, \qquad x \in \mathbb{R}.$$

This equation appears as a cubic perturbation of the Schrödinger equation for the wave function of a free one-dimensional particle – whence the name. However, its physical meaning goes far beyond one-particle quantum mechanics.

Among others, the NLS equation describes slowly varying wave envelopes in dispersive media and arises in various physical systems such as water waves, plasma physics, solid-state physics and nonlinear optics. One of the most successful applications of the NLS equation is the description of the propagation of optical solitons in fibers – see for example [2] and the references therein.

After the KdV equation, the cubic NLS equation was the second equation which was discovered to be integrable by the inverse scattering method [48]. It turned out that it has the same degree of universality as the KdV equation, both from a mathematical and a physical point of view. Even more, in many technical aspects the NLS equation is simpler. For instance, its Hamiltonian formalism is the standard one, while the formalism for KdV involves a partial derivative. Moreover, the NLS equation is also considered in higher dimensions and plays an important role in quantum mechanics.

These remarks also apply to the *focusing nonlinear Schrödinger equation*,

$$\mathrm{i}\partial_t u = -\partial_x^2 u - 2\,|u|^2 u.$$

Typically, both these equations are studied either on the real line or on the circle, that is, with periodic boundary conditions

$$u(x+1,t) = u(x,t), \qquad t \in \mathbb{R}.$$

These four cases are actually quite different from each other and need to be studied separately.

In the following we consider the *defocusing nonlinear Schrödinger equation on the circle*. Our aim is to provide a complete and self-contained study of this evolution equation as a Hamiltonian system. In particular, we will construct a global coordinate system, in which the NLS equation appears as a classical integrable Hamiltonian system with infinitely many degrees of freedom.

Hamiltonian formalism The NLS equation is a Hamiltonian PDE. More precisely, it can be written in a Hamiltonian form, which encompasses both the focusing and defocusing case. Let

$$H_{\mathbb{C}}^m := H^m(\mathbb{T},\mathbb{C}), \qquad m \geqslant 1,$$

denote the Hilbert space of all complex valued functions on the circle $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ with m distributional derivatives in L^2. Let

$$H_c^m := H_{\mathbb{C}}^m \times H_{\mathbb{C}}^m$$

denote the phase space with elements $\varphi = (\varphi_1, \varphi_2)$. The associated Poisson bracket is given by

$$\{F, G\} := -\mathrm{i} \int_{\mathbb{T}} (\partial_{\varphi_1} F\, \partial_{\varphi_2} G - \partial_{\varphi_2} F\, \partial_{\varphi_1} G)\, \mathrm{d}x,$$

where $\partial_{\varphi_1} F$ and $\partial_{\varphi_2} F$ denote the components of the gradient $\partial_\varphi F$ of a C^1-functional F with respect to the standard L^2-product.

The NLS Hamiltonian

$$H_{\mathrm{NLS}} = \int_{\mathbb{T}} (\partial_x \varphi_1 \partial_x \varphi_2 + \varphi_1^2 \varphi_2^2)\, \mathrm{d}x$$

then gives rise to the Hamiltonian equations of motion

$$\begin{aligned} \mathrm{i}\partial_t \varphi_1 &= \partial_{\varphi_2} H_{\mathrm{NLS}} = -\partial_{xx}\varphi_1 + 2\varphi_2\varphi_1^2, \\ \mathrm{i}\partial_t \varphi_2 &= -\partial_{\varphi_1} H_{\mathrm{NLS}} = \partial_{xx}\varphi_2 - 2\varphi_1\varphi_2^2. \end{aligned} \tag{$*$}$$

The defocusing NLS equation is obtained by restricting this system to the invariant subspace

$$H_r^m = \{\varphi \in H_c^m : \varphi_2 = \bar{\varphi}_1\}$$

of states φ of *real type*. Taking $\varphi = (u, \bar{u})$, we get

$$H_{\mathrm{NLS}} = \int_{\mathbb{T}} (|\partial_x u|^2 + |u|^4)\, \mathrm{d}x$$

for the restricted Hamiltonian, and the equations of motion reduce to

$$\mathrm{i}\partial_t u = \mathrm{i}\,\{u, H\} = \partial_{\bar{u}} H$$

for $H = H_{\mathrm{NLS}}$, familiar from classical mechanics.

The focusing NLS equation, on the other hand, is obtained as the restriction of the general equation to the invariant subspace

$$H_i^m = \{\varphi \in H_c^m : \varphi_2 = -\bar{\varphi}_1\}$$

of states φ of *imaginary type*.

The above Hamiltonians are well defined on H_c^m only when $m \geqslant 1$. On the other hand, the initial value problem for the NLS equation on the circle is well posed on

$$L_c^2 := L_{\mathbb{C}}^2(\mathbb{T}, \mathbb{C}) \times L_{\mathbb{C}}^2(\mathbb{T}, \mathbb{C}) = H_c^0$$

as well – see [9]. This will also be the setting of the global coordinates to be constructed in the sequel.

Lax pair formalism Based on the seminal work of Gardner et al. [16] and Lax [34] for the KdV equation, Zakharov and Shabat [48] discovered a Lax pair for the NLS equation and showed in this way that it admits infinitely many integrals in involution – see also [1, 13, 35] and the references therein.

More precisely, consider for $\varphi = (\varphi_1, \varphi_2)$ in L^2_c the *Zakharov-Shabat* or ZS-*operator*

$$L(\varphi) = \begin{pmatrix} \mathrm{i} & 0 \\ 0 & -\mathrm{i} \end{pmatrix} \frac{\mathrm{d}}{\mathrm{d}x} + \begin{pmatrix} 0 & \varphi_1 \\ \varphi_2 & 0 \end{pmatrix}.$$

We call φ the *potential* of the operator $L(\varphi)$.

Suppose φ also depends differentiably on time t, giving rise to a family of operators

$$L(t) := L(\varphi(t, \cdot)).$$

Then, by a tedious but elementary calculation, φ is a solution of the NLS equation $(*)$ if and only if

$$\partial_t L = [B, L]\,,$$

where $[B, L] = BL - LB$ denotes the commutator of L with the operator

$$B = \begin{pmatrix} 2\mathrm{i}\partial_x^2 - \mathrm{i}\varphi_1\varphi_2 & \varphi_1' + 2\varphi_1\partial_x \\ \varphi_2' + 2\varphi_2\partial_x & -2\mathrm{i}\partial_x^2 + \mathrm{i}\varphi_1\varphi_2 \end{pmatrix}.$$

On the subspaces H^m_r of potentials of real type, B is formally skew-adjoint, that is, $B^* = -B$. In this case the flow of

$$\partial_t V = BV, \qquad V(0) = I$$

generates, at least formally, a one-parameter-family of unitary operators $V(t)$, since

$$\partial_t(V^*V) = (\partial_t V^*)V + V^*\partial_t V = V^*(B^* + B)V = 0.$$

Moreover, by an analogous calculation one finds that

$$\partial_t(V^*LV) = V^*(\partial_t L - [B, L])V = 0,$$

whence

$$V^*(t)L(t)V(t) = L(0), \qquad t \in \mathbb{R}.$$

The spectrum of $L(t)$ is thus independent of t. Put differently, the flow of the defocusing NLS equation defines an *isospectral deformation* on the space of all potentials of real type. The whole space L^2_r decomposes into *isospectral sets*

$$\mathrm{Iso}(\varphi) = \left\{\psi \in L^2_r : \mathrm{spec}(\psi) = \mathrm{spec}(\varphi)\right\}, \qquad \varphi \in L^2_r,$$

which are *invariant* under the defocusing NLS flow.

Spectrum When φ is of real type – as in the defocusing case – then $L(\varphi)$ is formally self-adjoint. As is well known – and will be proven in detail in section II – the spectrum of $L(\varphi)$ considered on the interval $[0,2]$ with periodic boundary conditions is pure point and consists of an unbounded bi-infinite sequence of periodic eigenvalues

$$\cdots < \lambda_{-1}^{-} \leqslant \lambda_{-1}^{+} < \lambda_0^{-} \leqslant \lambda_0^{+} < \lambda_1^{-} \leqslant \lambda_1^{+} < \cdots \quad .$$

The – possibly empty – intervals $(\lambda_n^-, \lambda_n^+)$ are called the spectral *gaps* of the potential φ, and

$$\gamma_n = \lambda_n^+ - \lambda_n^- \geqslant 0, \qquad n \in \mathbb{Z},$$

the corresponding *gap lengths*.

By the way, the complementary intervals $[\lambda_n^+, \lambda_{n+1}^-]$ are called the spectral *bands* of φ, but we will not make use of them.

By the Lax pair formalism, each periodic eigenvalue is an integral of motion. From an analytical point of view, however, these integrals are not satisfactory, as $\lambda_n^{\pm}$ is not a smooth function of φ whenever the correponding gap collapses. The *squared gap lengths*

$$\gamma_n^2 = (\lambda_n^+ - \lambda_n^-)^2, \qquad n \in \mathbb{Z},$$

however, are *analytic functions* on all of L_r^2 and thus form a usable set of integrals. Moreover, they determine the periodic spectrum, so that

$$\begin{aligned} \operatorname{Iso}(\varphi) &= \left\{ \psi \in L_r^2 \colon \operatorname{spec}(\psi) = \operatorname{spec}(\varphi) \right\} \\ &= \left\{ \psi \in L_r^2 \colon (\gamma_n(\psi))_{n\in\mathbb{Z}} = (\gamma_n(\varphi))_{n\in\mathbb{Z}} \right\}. \end{aligned}$$

These sets are compact connected tori whose dimension equals the number of positive gap lengths and is infinite generically. They are called Lagrangian with respect to the Poisson structure defined above when all spectral gaps are open.

A Hamiltonian PDE with the property that its invariant sets are generically Lagrangian tori is often referred to as an *integrable* PDE.

Normal form In classical mechanics the existence of a foliation of the phase space into Lagrangian invariant tori is tantamount, at least locally, to the existence of action-angle coordinates. This is the content of the Liouville-Mineur-Arnold-Jost theorem. In an infinite-dimensional setting, the existence of such coordinates is far less clear, as the dimension of the foliation is nowhere locally constant. Invariant tori of infinite and finite dimension each form dense subsets. Nevertheless, action-angle coordinates can be introduced for the NLS equation, as we describe now.

To state the result, we introduce the spaces of real sequences

$$h_r^m := \{ (x,y) = (x_n, y_n)_{n\in\mathbb{Z}} \colon \|x\|_m + \|y\|_m < \infty \}, \qquad m \geqslant 0,$$

where

$$\|x\|_m^2 = x_0^2 + \sum_{n\in\mathbb{Z}} n^{2m} x_n^2.$$

When $m = 0$ we also write

$$\ell_r^2 := h_r^0.$$

The space h_r^m is equipped with the Poisson structure induced by the canonical symplectic structure

$$\omega = \sum_{n \in \mathbb{Z}} \mathrm{d}x_n \wedge \mathrm{d}y_n.$$

Theorem. *There exists a bi-analytic diffeomorphism $\Omega\colon L_r^2 \to \ell_r^2$ with the following properties:*

1. *Ω is canonical, that is, preserves Poisson brackets.*
2. *The restriction of Ω to H_r^m with $m \geqslant 1$ gives rise to a map $\Omega\colon H_r^m \to h_r^m$ that is again onto and bi-analytic.*
3. *Ω introduces global Birkhoff coordinates for* NLS *on H_r^1. That is, on h_r^1 the transformed* NLS *Hamiltonian $H_{\mathrm{NLS}} \circ \Omega^{-1}$ is a real-analytic function of the actions $I_n = \left(x_n^2 + y_n^2\right)/2$ with $n \in \mathbb{Z}$.*
4. *$d_0\Omega$ is the Fourier transform.*

Actually, we prove a more general version of this theorem, where we consider the restrictions of Ω to various weighted Sobolev spaces. Moreover, the map Ω introduces Birkhoff coordinates for every Hamiltonian in the Poisson algebra consisting of Hamiltonians commuting with all actions I_n. This applies, in particular, to all Hamiltonians in the NLS hierarchy.

The construction of Birkhoff coordinates for a potential $\varphi \in L_r^2$ actually starts out with the definition of candidates for the actions I_n and angles θ_n. Those are defined in terms of certain path integrals on the two-sheeted complex curve associated with the periodic spectrum of $L(\varphi)$. No reference to the NLS equation is required for this construction. The I_n are defined on L_r^2, while each θ_n is defined on the dense open subset of potentials where $\gamma_n \neq 0$. The details are explained in Section 11.

Denoting the transformed NLS Hamiltonian by the same symbol, we thus obtain a real analytic Hamiltonian

$$H_{\mathrm{NLS}} = H_{\mathrm{NLS}}(\dots, I_{-1}, I_0, I_1, \dots)$$

on h_r^1. Its equations of motion are the classical ones,

$$\dot{x}_n = \omega_n(I) y_n, \qquad \dot{y}_n = -\omega_n(I) x_n,$$

where $\omega_n(I) = \partial_{I_n} H_{\mathrm{NLS}}(I)$ are the NLS frequencies, determined by the initial values.

Historical comments We conclude this overview with a few of historical comments concerning the construction of action-angle variables.

The angle variables θ_n for integrable PDEs such as KdV and NLS were introduced for finite-gap potentials (introduced at the end of Section 9) in the early 1970s by Dubrovin, Kričever, and Novikov [12] and McKean and van Moerbecke [38], and further investigated, among others, by McKean and Trubowitz [39, 40], Its and Matveev [22], and Belokolos et al. [8]. These authors used elements of Riemann surface theory to show that the θ_n linearize PDEs such as KdV and NLS. In this way they obtained quasi-periodic solutions of these equations. For later work along these lines see for instance [14, 43, 45].

The formulas for the action variables I_n were first presented in the case of KdV and the Toda lattice by Flaschka and McLaughlin [15]. They were obtained by following Arnold's approach of defining actions and based on a system of canonical coordinates involving the Dirichlet eigenvalues. Their construction was later generalized by Veselov and Novikov [46].

In the early 1990s Birkhoff coordinates were constructed for the periodic KdV equation [23, 4] as well as the defocusing NLS equation [4, 5] and the Toda lattice [6, 20, 21]. The approach was inspired by Vey [47], who proved the existence of such coordinates for a finite-dimensional, integrable Hamiltonian system in a neighbourhood of an elliptic fixed point.

In the late 1990s McKean and Vaninsky [41, 42] showed that in the case of the defocusing NLS equation the actions and angles mentioned above actually Poisson commute. Subsequently we used their approach to provide a conceptually rather simple proof of the existence of Birkhoff coordinates for the KdV equation [25, 26], using concepts of Hamiltonian systems theory to prove that these variables lead to canonical coordinates defined on the entire phase space.