Preface

The subject of this monograph is the appearance of irreversibility in gas dynamics.

At a molecular level, the dynamics is Newtonian. In particular, it is reversible, in contrast with observations at a macroscopic level.

In 1872, Boltzmann introduced the equation

$$\text{(B)} \qquad \partial_t f + v \cdot \nabla_x f = Q(f, f),$$

where $x \in \mathbf{R}^d$ represents position and $v \in \mathbf{R}^d$ velocity, for the probability density $f(t, x, v)$ known as the distribution function of the gas. The bilinear collision operator Q is related to a jump process in the velocity variable. The dynamics of the Boltzmann equation locally preserves mass, momentum and energy, as does the Newtonian microscopic dynamics. In addition, the Boltzmann equation admits a Lyapunov functional, known as the entropy, which is nondecreasing along trajectories. This is a feature of an irreversible dynamics.

The specific question that we address in this monograph is the relationship between the Newton dynamics for a system of particles and the Boltzmann dynamics.

A partial answer is given in Oscar Lanford's 1975 theorem [35], which accounts for some important intuitions of Boltzmann [9]:

- equation (B) should be obtained as a limit when the number of particles becomes large. In Boltzmann's words: *The velocity distribution of the molecules is not mathematically exact as long as the number of molecules is not assumed to be mathematically infinitely large.*
- equation (B) predicts only the most probable behavior. In particular, it does not account for trajectories along the Newtonian flow which have decreasing entropy: *In nature, the tendency is to pass from the least likely state to the more likely.* [...] *The second principle in Thermodynamics appears therefore as a probability theorem.*
- a central question in the derivation of equation (B) is the independence of elementary particles : *From now on we shall specifically assume that the motion is totally disorganized, either as an ensemble or at a molecular level, and that it remains so indefinitely.*

Lanford's theorem states that the distribution function of a system of N particles, which are interacting with one another by elastic collisions and are initially independent and smoothly distributed, converges to the solution of the Boltzmann equation (B) in the limit $N \to \infty$, if the characteristic length of interaction ε simultaneously goes to 0 in the Boltzmann-Grad scaling limit $N\varepsilon^{d-1} = O(1)$.

A striking point in Lanford's theorem is that it partially invalidates the third intuition of Boltzmann: the independence is rigorously established in the limit, under the mere assumption that it holds initially.

The main limitation in the theorem is that the convergence is proved to hold only on small time intervals, in which typically only a small number of collisions per particle take place.

As we shall see, trajectories that are not accounted for in the Boltzmann dynamics involve recollisions, meaning interactions between particles which have previously interacted in the past (directly or indirectly). Such trajectories violate independence. The strategy of Lanford was then to decompose the dynamics in terms of collision trees and prove that

- with probability converging to 1, collisions trees are finite, and
- with probability converging to 1, recollisions do not happen in finite trees.

It seems however that the arguments used in the literature to establish the second point were not entirely correct, so that at some point the proof should be completed.

The aim of this monograph is to provide such a completion of the proof of Lanford's theorem, in a self-contained manner. In addition, building on the important contribution of King [31], the convergence result is extended to systems of particles interacting pairwise via compactly supported potentials satisfying a convexity assumption. We also discuss in depth the notion of independence. In the hard-sphere case, precise bounds in all steps of the proof enable us to obtain a rate of convergence.

We insist on the fact that the strategy of the proof is by no means new. The main novelty here is the detailed study of trajectories involving recollisions. This is the key point that allows to prove the term-by-term convergence result in the correlation series expansion.

Part I gives some context: we discuss low-density limits, recall some of the main landmarks in the vast literature concerning the Boltzmann equation, and state the main theorems proved in this monograph.

In Part II we focus on the hard-sphere case. We first derive the BBGKY hierarchy associated with the Liouville equation, and prove that it is well-posed on a short time interval, uniformly in the number of particles. Then we turn to the notion of independence, which is central in Lanford's theorem. Finally we give a precise convergence statement of the BBGKY hierarchy to the Boltzmann hierarchy. The convergence to the Boltzmann equation then appears as the particular case of tensor products. We finally present the salient features of the proof.

Part III is devoted to the case of particle interactions produced by a compactly supported potential. We first study the scattering operator associated with two-particle interactions, and then derive the associated BBGKY hierarchy. This derivation is rendered delicate by the fact that simultaneous interactions of large numbers of particles may occur. Only pairwise interactions contribute to the dynamics in the limit, however, and bounds similar to the ones in the hard-sphere case are derived. A precise statement of convergence towards the limiting Boltzmann hierarchy is given, and a strategy of proof is presented.

Part IV presents the proofs of both convergence results (hard spheres and short-range potential). The fact that potential interactions are non-local produces only minor differences between the proofs. The study of trajectories involving recollisions, which deviate substantially from the Boltzmann trajectories, is performed in detail.

In particular, we provide explicit (semi-explicit, in the case of a potential) bounds on their size. As a consequence, in the hard-sphere case a rate of convergence can be obtained. A list of open problems concludes the text.

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