Contents

1 Introduction . . . 1
1.1 From the Ginzburg–Landau model to the 2D Coulomb gas . . . 1
1.1.1 Superconductivity and the Ginzburg–Landau model . . . 1
1.1.2 Reduction to a Coulomb interaction . . . 3
1.2 The classical Coulomb gas . . . 5
1.2.1 The general setting . . . 5
1.2.2 Two-dimensional Coulomb gas . . . 6
1.2.3 The one-dimensional Coulomb gas and the log gas . . . 8

2 The leading order behavior of the Coulomb gas . . . 11
2.1 Γ-convergence: general definition . . . 12
2.2 Γ-convergence of the Coulomb gas Hamiltonian . . . 14
2.3 Minimizing the mean-field energy via potential theory . . . 15
2.4 The mean field limit . . . 23
2.5 Linking the equilibrium measure with the obstacle problem . . . 29
2.5.1 Short presentation of the obstacle problem . . . 29
2.5.2 Connection between the two problems . . . 32
2.6 Large deviations for the Coulomb gas with temperature . . . 35

3 Splitting the Hamiltonian . . . 43
3.1 Expanding the Hamiltonian . . . 43
3.2 The truncation procedure and splitting formula . . . 46
3.3 The case $d = 1$ . . . 50
3.4 Almost monotonicity . . . 53
3.5 The splitting lower bound . . . 54
3.6 Consequences . . . 55
3.7 Control of the potential and charge fluctuations . . . 56

4 Definition(s) and properties of renormalized energy . . . 61
4.1 Motivation and definitions . . . 61
4.2 First properties . . . 65
4.3 Almost monotonicity of $\mathcal{W}_\eta$ and lower bound for $\mathcal{W}$ . . . 68
4.4 Well separated and periodic configurations . . . 70
4.5 Partial results on the minimization of W and $\mathcal{W}$, and the crystallization conjecture . . . 76

5 Deriving $\mathcal{W}$ as the large n limit: lower bound via a general abstract method . . . 81
5.1 Lower bound for 2-scales energies . . . 81
5.2 Next order for the Coulomb gas . . . 87

5.2.1 Assumptions 87
5.2.2 Lower bound 87

6 Deriving $\mathcal{W}$ as the large n limit: screening, upper bound, and consequences 93
6.1 Separation of points and screening 93
6.2 Upper bound and consequences for ground states 99
6.3 Consequences on statistical mechanics 103

7 The Ginzburg–Landau functional: presentation and heuristics 107
7.1 The functional 107
7.2 Types of states, critical fields 108
7.2.1 Types of solutions and phase transitions 108
7.2.2 Vortex solutions 109
7.2.3 Related models 110
7.3 Heuristics 111
7.3.1 Rough heuristics 111
7.3.2 The vorticity measure and the London equation 112
7.3.3 Formal derivation of the first critical field 113

8 Main mathematical tools for Ginzburg–Landau 117
8.1 The ball construction method 117
8.1.1 A sketch of the method 118
8.1.2 A final statement 122
8.2 The “Jacobian estimate” 124

9 The leading order behavior for Ginzburg–Landau 127
9.1 The Γ-convergence result 127
9.2 The proof of Γ-convergence 128
9.2.1 Lower bound 128
9.2.2 Upper bound 130
9.3 Mean-field limit and obstacle problem 133
9.4 The intermediate regime near H_{c_1} 135

10 The splitting and the next order behavior for Ginzburg–Landau 137
10.1 Splitting 137
10.2 Deriving W from Ginzburg–Landau 141
10.2.1 Rescaling and notation 141
10.2.2 Lower bound 142
10.2.3 Upper bound 145
10.2.4 A statement of main result 146

Bibliography 147

Index 157