Introduction

There is no universal agreement on what constitutes a "good" mathematical problem. One possible measurement could be the amount of interesting mathematics that it leads to. From that point of view, the search for (closed) geodesics, originating with the works of Hadamard [38] and Poincaré [53] and with substantial early contributions by Birkhoff [8], Morse [52], and Lyusternik and Schnirel'man [48] certainly qualifies. The general problem is to find geodesics connecting two given points of a Riemannian manifold or to find periodic geodesics, and to give a meaning to their count. The most important offspring of this problem is the development of topological methods in variational calculus, generally referred to as Morse theory (or, as Bott puts it, "Morse theory indomitable" [9]). One of its most recent incarnations is Floer theory, a central tool in modern symplectic topology. The geodesic problem also led to the development of computational tools in algebraic topology (spectral sequences), and is connected to the theory of minimal models and to Hochschild and cyclic homology.

In attempts to solve the geodesic problem one is quickly led to the study of spaces of paths and loops on manifolds. These spaces have been the object of much interest in recent years, both for topologists and symplectic geometers. The main purpose of this book is to facilitate communication between these two communities by developing a common basis. From a topological point of view, a lot is known about path and loop spaces, but the results are often scattered throughout the literature. In particular, it can be difficult for a newcomer to the subject to extract the main lines of thought. Hopefully, this book will serve as a guide to these topological techniques and results. At the same time, the symplectic point of view emphasizes certain features and algebraic structures that have been little or not studied at all. This relates in particular to the modern development known as string topology. It seems reasonable to expect that questions from symplectic topology will motivate new developments in the topological study of free loop spaces, and conversely.

Genesis of this book

This book grew out of a learning seminar on "Free loop spaces" held at Strasbourg University in 2008–2009. The seminar attracted a much bigger audience than initially expected, and many of the speakers agreed to expand their talks into chapters for this book. The guiding rule that we tried to follow was to keep the level of exposition accessible to a graduate student. Our goal in this book is not to present the latest developments, but rather to build from the basics up to some level from which the interested reader could continue on her or his own. Some of the chapters also contain new research material, most notably the one by Hossein Abbaspour.

The contribution that stands out particularly is Mohammed Abouzaid's "Symplectic cohomology and Viterbo's theorem", which constitutes Part II of this book. Though initially intended as one of the chapters, it grew into a fully fledged research monograph. It does start gently by discussing some foundational facts from symplectic geometry, and also from the Morse theory of finite-dimensional approximations of free loop spaces. But rather than sketching one of the published proofs of Viterbo's result, it then proceeds to give a new proof, building on ideas of previous approaches but developing an original point of view.

Unfolding the story

Let us now proceed with a more detailed description of the story that is told in this book. Our aim here is not so much to offer a strictly historical perspective – though we do provide some historical background – but rather to introduce the mathematical subjects and objects discussed in the book.

We denote $S^1 = \mathbb{R}/\mathbb{Z}$ and, for a manifold M, we denote its free loop space by

$$\mathcal{L}M := \{\gamma : S^1 \to M : \gamma \text{ continuous}\} .$$

In the subsequent discussions one sometimes has to consider subspaces of $\mathcal{L}M$ consisting of loops that satisfy additional regularity properties, but in this introduction we will use the uniform notation $\mathcal{L}M$.

Riemannian geometry. Given a Riemannian manifold (M, g) the closed geodesics parametrized by S^1 are the critical points of the energy functional

$$E : \mathcal{L}M \to \mathbb{R}, \quad E(\gamma) := \frac{1}{2}\int_{S^1} \|\dot{\gamma}(t)\|^2 \, dt .$$

Here the most convenient setup is that of loops of Sobolev class H^1. With this choice, the energy functional is well-behaved in several respects: (i) it is bounded from below, (ii) it satisfies the Palais–Smale condition (any sequence γ_k such that $E(\gamma_k)$ is bounded and $\|\nabla\gamma_k\| \to 0$ has a convergent subsequence), (iii) for a generic metric the critical set is a disjoint union of submanifolds (one copy of M that corresponds to the constant loops and a countable union of disjoint circles given by nontrivial geodesics and their shift reparametrizations), and (iv) the Hessian d^2E is non-degenerate and has finite index in the normal direction to any critical submanifold (we say that E is Morse–Bott). Morse theory is designed to handle precisely this kind of functionals. The outcome is a description of the loop space $\mathcal{L}M$ by successive attachments of bundles over the critical submanifolds with rank given by the index of d^2E. This allows a grip on the topology of $\mathcal{L}M$ provided one has enough information on these indices and on the attaching maps. Conversely, knowledge of the topology of $\mathcal{L}M$ implies existence results for critical points of E.

One significant difficulty in converting existence results for critical points of E into existence results for geometrically distinct closed geodesics is that every nonconstant closed geodesic can be iterated and hence gives rise to countably many distinct critical submanifolds. Still, in many cases this problem can be overcome. The most powerful result in this direction is the following theorem due to Gromoll–Meyer.

Theorem (Gromoll–Meyer [35]). *Let M be a simply connected closed manifold such that the sequence $\{b_k(\mathcal{L}M)\}$, $k \geq 0$ of Betti numbers of $\mathcal{L}M$ with coefficients in some field is unbounded. Then for any Riemannian metric on M there exist infinitely many geometrically distinct closed geodesics.*

The difficult content of the theorem is that the conclusion holds for *any* metric, not only for a generic one. Thus the critical set of E is not assumed to be well-behaved, and so the proof needs ideas beyond the Morse theory picture sketched above. This theorem and related ideas from the calculus of variations are discussed in Chapter 2 of this book. To make effective use of this result, one needs to know when its topological assumptions hold, and this brings us to the discussion in the next section.

Minimal models. The starting point for the topological study of the free loop space $\mathcal{L}M$ is the loop-loop fibration (see Chapter 1)

$$\begin{array}{ccc} \Omega M & \longrightarrow & \mathcal{L}M \\ & & \big\downarrow{\scriptstyle ev} \\ & & M \end{array}$$

Here ev is the evaluation at the origin of a loop, and ΩM is the based loop space, consisting of loops starting and ending at a fixed basepoint in M. This fibration can be used to determine the homotopy groups of $\mathcal{L}M$, namely $\pi_k(\mathcal{L}M) \simeq \pi_k(M) \oplus \pi_k(\Omega M)$: indeed, the section given by the inclusion of constant loops determines a splitting of the homotopy long exact sequence (Chapter 1). However, the situation is very different as far as homology groups are concerned. It turns out that the Leray–Serre spectral sequence is effective in simple cases (spheres [50, 21]) but of very limited use in general, unless one has additional geometric information about the differentials. Indeed, the path-loop fibration with the same fiber ΩM and the same base M has contractible total space, so any successful reasoning must use specific features of the loop-loop fibration.

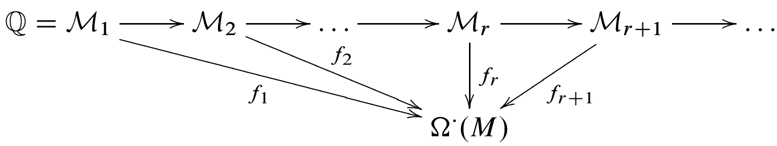
A general solution for the computation of $H^{\cdot}(\mathcal{L}M;\mathbb{Q})$ (and hence of $H_{\cdot}(\mathcal{L}M;\mathbb{Q})$) for simply connected spaces M was made available by Sullivan's theory of *minimal models* ([57], see also [56]). It turns out that this theory is powerful enough to clarify exactly when the assumptions of the Gromoll–Meyer theorem hold.

Theorem (Sullivan–Vigué-Poirrier [58]). *Let M be a simply connected closed manifold. The sequence of Betti numbers of $\mathcal{L}M$ with coefficients in $\mathbb{Q}$ is unbounded if and only if $H^{\cdot}(M;\mathbb{Q})$ requires at least two generators as a ring.*

This theorem and the theory of minimal models are explained in Luc Menichi's Chapter 3.

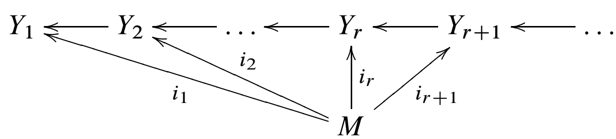
A *minimal model* for a commutative differential graded algebra (cdga) $A^\cdot$ over a field $\mathbf{k}$ is a cdga $(\mathcal{M}^\cdot, d)$ with a quasi-isomorphism $\mathcal{M}^\cdot \to A^\cdot$ such that $\mathcal{M}^\cdot$ is free (i.e. $\mathcal{M}^\cdot = \Lambda V^\cdot$ with $V^\cdot$ a graded vector space) and its differential satisfies $d(\mathcal{M}^\cdot) \subseteq \mathcal{M}^+ \cdot \mathcal{M}^+$, with $\mathcal{M}^+ = \oplus_{k \geq 1} \mathcal{M}^k$. We call $\mathcal{M}^\cdot$ a "model" because it is quasi-isomorphic to $A^\cdot$ and it is free as an algebra (thus, the complexity of the algebra structure of $A^\cdot$ has been moved into the differential for $\mathcal{M}^\cdot$). We call $\mathcal{M}^\cdot$ "minimal" because it contains no unnecessary generators, which is expressed by the condition $d(\mathcal{M}^\cdot) \subseteq \mathcal{M}^+ \cdot \mathcal{M}^+$. There always exists a minimal model provided $H^0 = \mathbf{k}$; when a minimal model exists, it is unique up to isomorphism [23, Thm. 14.12].

The construction of a (real) minimal model $\mathcal{M}$ for a simply connected manifold M proceeds inductively, starting from the de Rham algebra of differential forms $(\Omega^\cdot(M), \wedge, d)$ and building a sequence



where $\mathcal{M}_r$ is built from $\mathcal{M}_{r-1}$ by adjoining generators of degree r in such a way that the map f_r induces an isomorphism on cohomology in degrees $\leq r$ and is injective in degree $r+1$, and the horizontal maps are inclusions. One of Sullivan's insights was that over $\mathbb{Q}$ one can still build a cdga $(A^\cdot, d)$ of rational differential forms, so that the same construction applies and builds a *rational* minimal model for M. The existence of Steenrod operations shows that there is no hope of extending this discussion to integer coefficients, since their construction is directly based on the *failure of commutativity* of the cup product on the chain level over $\mathbb{Z}$ (see e.g. [55, §5.9] or [40, §4.L] for the construction of Steenrod squares).

The construction of minimal models is an algebraic analogue (in a way which is discribed e.g. by Félix, Oprea and Tanré in [24, §2.5.4], see also Bott–Tu [10] or Sullivan [56] for intuitive discussions) of the Postnikov tower of the manifold M, which is a sequence



where the maps $i_r \colon M \to Y_r$ are inclusions that induce isomorphisms of homotopy groups up to degree r, and all homotopy groups of Y_r in dimensions larger than r vanish. Moreover, each map $Y_r \to Y_{r-1}$ is a fibration whose fiber is an Eilenberg–MacLane space $K(\pi_r(M), r)$.

Exploiting this point of view, one finds that the minimal model $\mathcal{M}^{\cdot} = \Lambda V^{\cdot}$ of a simply connected manifold M satisfies the isomorphism (see Chapter 3, §3)

$$V^q \simeq \mathrm{Hom}(\pi_q(M) \otimes_{\mathbb{Z}} \mathbb{Q}, \mathbb{Q}) \,, \tag{1}$$

hence the role played by minimal models in rational homotopy theory [23, 24].

Returning to our story, the point is that, given a (rational) minimal model for a manifold M, there is an easy and explicit formula to obtain a (rational) minimal model for its free loop space [58]. The algebraic construction is described in section 4 of Luc Menichi's Chapter 3, while the intuition is derived from considering the adjunction

$$\mathrm{Map}(K, \mathcal{L}M) \cong \mathrm{Map}(K \times S^1, M)$$

for compact spaces K [56, 58]. Thus cohomological properties of the manifold translate into cohomological properties of its free loop space, and this circle of ideas leads to the proof of the Sullivan–Vigué-Poirrier theorem.

Hochschild and cyclic homology. The problem of computing the homology groups $H_{\cdot}(\mathcal{L}M; \mathbb{Q})$ for a closed simply connected manifold M is solved via Sullivan's theory of minimal models. We now introduce a completely different point of view which relates to Hochschild and cyclic homology.

Hochschild homology initially appeared in the study of deformation theory of associative algebras [42, 29], whereas cyclic homology is a more recent theory that was discovered by Connes in relation with non-commutative geometry [17]. A standard reference is Loday's book [46]. We use below the notation $HH_{\cdot}(A, A)$ and $HH^{\cdot}(A, A)$ for the Hochschild homology/cohomology of a differential graded algebra (dga) A, and $HC_{\cdot}(A, A)$, $HC^{\cdot}(A, A)$ for their cyclic counterparts. These algebraic objects are described and studied from various points of view in the chapters by Abbaspour, Menichi, Loday, and Félix.

There are two relevant theorems for our purpose of understanding free loop spaces.

Theorem (Burghelea–Fiedorowicz, Goodwillie [11, 33]). *Given a manifold M denote $S_{\cdot}(\Omega M)$ the strictly associative algebra of singular chains on the Moore loop space with the Pontryagin product. We have canonical isomorphisms*

$$HH_{\cdot}(S_{\cdot}(\Omega X)) \cong H_{\cdot}(\mathcal{L}X)$$

and

$$HC_{\cdot}(S_{\cdot}(\Omega X)) \cong H_{\cdot}^{S^1}(\mathcal{L}X) \,.$$

Here $H_{\cdot}^{S^1}(\mathcal{L}X)$ denotes the S^1-equivariant homology of the free loop space $\mathcal{L}X$, viewed as an S^1-space where the action rotates the domain circle. By definition $H_{\cdot}^{S^1}(\mathcal{L}X)$ is the homology of the homotopy quotient $\mathcal{L}X \times_{S^1} ES^1$, where ES^1 is the universal principal S^1-bundle (see Chapters 1 and 2). The Moore loop space is by definition the space of based loops parametrized by closed intervals of arbitrary length.

Theorem (Jones [43]). *Let M be a simply connected manifold and denote by $S^{\cdot}(M)$ the cdga of singular cochains on M. We have canonical isomorphisms*

$$HH^{\cdot}(S^{\cdot}(M), S^{\cdot}(M)) \cong H_{\cdot+\dim M}(\mathcal{L}M)\,,$$

and

$$HC^{\cdot}(S^{\cdot}(M), S^{\cdot}(M)) \cong H^{S^1}_{\cdot+\dim M}(\mathcal{L}M)\,.$$

Jones' theorem is discussed at length in the chapter by Loday, and an explicit example is worked out in the appendix by Latschev. As for the Burghelea–Fiedorowicz–Goodwillie isomorphisms, besides the original papers the reader can also consult [46, §7.3], as well as the sketch of proof in [45].

The importance of these constructions is that Hochschild *cohomology* of any algebra has the structure of a Gerstenhaber algebra [29]. In some relevant cases (e.g. cochains on a smooth closed oriented manifold with coefficients in a field of characteristic zero), this Gerstenhaber algebra structure lifts to a Batalin–Vilkovisky (BV) algebra structure [60, 25], and BV structures are relevant because they are algebraic incarnations of S^1-actions. These algebraic structures and their interplay are studied at length from the broad perspective of Calabi–Yau algebras in Chapter 6 by Hossein Abbaspour. Some relations to (rational) string topology are also described in Chapter 7 by Yves Félix, which brings us to our next topic.

String topology. In 1999 Chas and Sullivan [14] discovered a new and fundamental piece of structure on the homology $H_{\cdot}(\mathcal{L}M)$, namely an associative product, called *loop product* or *Chas–Sullivan product*. This product is discussed in the book from various perspectives by Chataur and Oancea, Félix, Abbaspour, and Abouzaid.

Recall the loop-loop fibration $\Omega M \hookrightarrow \mathcal{L}M \to M$. The based loop space ΩM has a natural H-space structure from concatenating loops, and so its homology inherits the so called Pontryagin product. Intuitively, the loop product on $H_{\cdot}(\mathcal{L}M)$ ties together the Pontryagin product on $H_{\cdot}(\Omega M)$ and the intersection product on the base manifold M: given two cycles $\alpha, \beta \in C_{\cdot}(\mathcal{L}M)$ intersecting transversely one forms another cycle $\alpha\cdot\beta$ of degree $|\alpha|+|\beta|-\dim\ M$ by concatenating the elements of α with the elements of β whenever their corresponding basepoints coincide. In other words, one forms a fiber product of α and β over the evaluation maps, and then concatenates. This product interacts well with the degree 1 operation $\Delta\colon H_{\cdot}(\mathcal{L}M) \to H_{\cdot+1}(\mathcal{L}M)$ constructed from the S^1-action, called BV (for Batalin–Vilkovisky) operator. The resulting structure is summarized as follows.

Theorem (Chas–Sullivan [14]). *Let M be a closed oriented manifold. The (shifted) homology of its free loop space*

$$\mathbb{H}_{\cdot}(\mathcal{L}M) := H_{\cdot+\dim M}(\mathcal{L}M)$$

carries a natural BV-algebra structure, meaning the following : a graded commutative ring structure (loop product) and a degree 1 operator $\Delta\colon \mathbb{H}_{\cdot}(\mathcal{L}M) \to \mathbb{H}_{\cdot+1}(\mathcal{L}M)$ (action of the fundamental class of S^1), whose defect from being a graded derivation is a graded Lie bracket.

The precise definitions and statement can be found in §4 of Chapter 7 by Yves Félix. A Morse-theoretic model which also extends (with appropriate modifications) to nonorientable manifolds M is discussed in Chapter 3 of Mohammed Abouzaid's monograph.

Operations of a similar nature can also be defined on the S^1-equivariant homology groups $H_\cdot^{S^1}(\mathcal{L}M)$, and also on the cohomology groups $H^\cdot(\mathcal{L}M)$ and $H^\cdot_{S^1}(\mathcal{L}M)$. In the original paper [14] Chas and Sullivan construct a Lie bracket, the so-called *string bracket* on $H_\cdot^{S^1}(\mathcal{L}M)$, see also §5 of Chapter 7. Later [15] they upgraded this to the structure of an involutive Lie bialgebra on $H_\cdot^{S^1}(\mathcal{L}M, M)$.

In the special case of surfaces these structures on the S^1-equivariant homology of the loop space existed before the work of Chas and Sullivan, and in fact this special case was an important source of inspiration for the inception of string topology. Goldman [32, §5] defined a bracket on the linear span of free homotopy classes of closed oriented curves on a closed surface (the interesting case being that of genus $g \geq 2$). Intuitively, given two immersed and transverse representatives of such free homotopy classes, one concatenates them at each intersection point and considers the formal sum of the resulting free homotopy classes (which are in general distinct!). Goldman's construction underlies his Hamiltonian viewpoint on regular functions on character varieties. Turaev [61, §8.1] defined a cobracket on the quotient space of the same linear span of free homotopy classes of oriented curves by the 1-dimensional span of the trivial homotopy class and proved compatibility with the Goldman bracket, i.e. the bialgebra property. Turaev's construction underlies the fact that skein algebras of links in the cylinder lying over an oriented surface quantize the Lie algebra structure defined by Goldman.

Both these developments were specifically related to Teichmüller theory on the one hand and to the theory of knot and link invariants on the other hand. From this point of view, one can only wonder at Chas and Sullivan's marvelous discovery that the same kind of structure exists in higher dimensions.

String topology can be viewed as a topologist's interpretation of string theory: the fundamental constituents of the theory are not points, but rather loops, and these interact by merging together and forming other loops. From this point of view it is not surprising that in a more abstract language the resulting structure is governed by the framed little 2-discs operad and its generalizations. The importance of operads in the context of S^1-spaces was first observed by Getzler [30, 31]. Looking back, it seems surprising that it took so long until this kind of structure was discovered in the context of free loop spaces.

In some sense, the full mathematical implications of the ideas underlying string topology are far from understood. The original definition of the basic operations given by Chas and Sullivan involves various transversality considerations, and initially there was hope that string topology could be sensitive to the underlying smooth structure of the manifold. However, it was proved by [16, 19, 37] that the loop product and the string bracket are homotopy invariant. It is still conceivable that more refined versions of string topology are able to distinguish smooth structures. This is certainly one of the central questions in the field, and first steps in that direction have been taken by

Basu in his Ph.D. dissertation [5]. Also, the algebraic operations of string topology have only just started to find applications regarding the geodesic problem [41].

Free loop spaces and symplectic topology. A manifold is called *symplectic* if it is endowed with a smooth 2-form that is closed and non-degenerate. A classical theorem of Darboux states that symplectic manifolds are locally isomorphic to a ball in $\mathbb{R}^{2n}$ endowed with the standard symplectic form $\omega_{std} = \sum_{i=1}^{n} dx_i \wedge dy_i$. In particular, they all have even dimension, and their local behavior is completely determined by the properties of $(\mathbb{R}^{2n}, \omega_{std})$. The global topological study of symplectic manifolds is referred to as *symplectic topology*. One particularly important class of objects are the *Lagrangian submanifolds*, which are submanifolds of half-dimension on which the symplectic form vanishes.

The notion of a symplectic form has its roots in the geometric structure underlying classical mechanics (cf. the classical book of Arnol'd [4]). Indeed, in the modern Hamiltonian formulation of a conservative system, the phase space associated to a given configuration space Q is nothing else than its cotangent bundle T^*Q, endowed with the canonical symplectic form, given in local coordinates by the expression $\sum_i dp_i \wedge dq_i$.

One of the central examples in Hamiltonian dynamics has always been the N-body problem of celestial mechanics. Ever since Poincaré's pioneering work (discussed at length in [12], see in particular Chapters 6–8 there), special emphasis has been put on understanding periodic motions and their stability, because they form one of the keys with which it is sometimes possible to make inroads into the otherwise seemingly impenetrable complexity of these systems. In fact, Poincaré originally suggested to study the problem of closed geodesics as a "toy model" for the study of more general Hamiltonian systems. As with geodesics, periodic orbits generally can be seen as critical points of a functional, namely the *symplectic action functional*, which is defined on (a suitable version of) the free loop space.

At this point it may be useful to discuss analogies and differences between Riemannian geometry and symplectic geometry. We have gathered a few relevant notions from the two fields for comparison:

Riemannian metric	Symplectic form
Length	Area
Curve	Surface
Points	Curves
Energy	Energy
Geodesics	Harmonic maps
Minimizing geodesics	Holomorphic curves

Let us discuss the first half of the list. A Riemannian metric's main purpose is to measure the length of *curves*. A symplectic form's main purpose is to measure the area of *surfaces*. A popular viewpoint is to see curves as evolution lines of *points*: by analogy, surfaces can be seen as evolution lines of *loops*, or strings. From this perspective, loops or strings are to symplectic topology what points are to Riemannian geometry, and the free loop space, seen as the "moduli space of loops on a symplectic manifold", plays the role of the Riemannian manifold, which is the "moduli space of its own points".

As for the second half of the list, we already repeatedly mentioned that geodesics are singled out as critical points of the energy functional (see also Chapter 2). On the symplectic side, defining an energy functional on the space of maps with exi 2-dimensional source requires some additional data besides the symplectic form ω, which by itself is an object of too topological a nature. This piece of additional data is the choice of a (suitably compatible) almost complex structure J that makes the symplectic manifold M into an almost Kähler manifold, and the choice of a conformal structure j at the source Σ. Compatibility of ω and J can be formulated as the requirement that $g_J := \omega(\,.\,, J\,.\,)$ should be a Riemannian metric on M. Now the energy functional on the space of maps $u\colon \Sigma \to M$ is the associated L^2-energy with respect to this metric. The minimizers of the resulting energy functional are so-called *J-holomorphic curves*, i.e. maps $u\colon (\Sigma, j) \to (M, J)$ such that $du \circ j = J \circ du$ (see the monograph by Abouzaid). They are characterized by the fact that the energy

$$E(u) = \int_\Sigma u^*\omega$$

is a purely topological quantity.

J-holomorphic curves were introduced into symplectic geometry by Gromov in 1985 [36], and have been one of the central tools ever since. A few years later, Floer [26] invented his eponymous homology theory and in this way applied a variant of these curves in the study of the symplectic action functional of Hamiltonian dynamics. His theory exposes the close relation between the symplectic properties of the underlying manifold and the topology of the free loop space.

In this volume the reader can have a glimpse of two instances in which free loops and symplectic topology mutually illuminate each other. The first instance relates to the above point of view which sees loops as boundaries of J-holomorphic curves. Chapter 5 explains a general argument of Fukaya [28] that proves substantial restrictions on Lagrangian embeddings in $\mathbb{R}^{2n}$ with the standard symplectic form. First note that each moduli space of J-holomorphic discs with boundary on a Lagrangian submanifold $L \subset \mathbb{R}^{2n}$ determines a chain on the free loop space $\mathcal{L}L$, essentially by considering the restriction of each map $u\colon (D, \partial D) \to (\mathbb{R}^{2n}, L)$ to its boundary circle. These moduli spaces are indexed by relative homotopy classes in $\pi_2(\mathbb{R}^{2n}, L)$. Fukaya's important insight was that while these spaces are not compact, their compactification can be described in terms of the loop bracket of string topology. The core technical result can be stated in somewhat loose form as follows.

Theorem (Fukaya [28]). *Let L be a closed oriented spin Lagrangian submanifold of* $(\mathbb{R}^{2n}, \omega_{\mathrm{st}})$.

(i) There is a suitable chain level model of the loop bracket such that the moduli spaces of J-holomorphic discs with boundary on L give rise to chains in this model, which together form a Maurer–Cartan element.

(ii) The twisted differential associated to this Maurer–Cartan element is such that the cycle of constant loops becomes a boundary.

When combined with additional topological arguments, this theorem for example allows one to completely classify all irreducible closed oriented 3-manifolds which can be embedded as Lagrangian submanifolds in $\mathbb{R}^6$: they must be of the form $\Sigma \times S^1$ for some surface Σ.

The second instance relates to Hamiltonian dynamics on the phase space. It is well known that, given a Riemannian metric on a manifold Q, the Hamiltonian flow of the kinetic energy on T^*Q is equivalent to the geodesic flow on TQ, this being an instance of the Legendre transform. Viterbo proved [62] that the Floer cohomology of the kinetic energy, also called *symplectic cohomology of* T^*Q, is isomorphic over $\mathbb{Z}/2$ to $H_\cdot(\mathcal{L}Q)$ in case Q is closed. Viterbo's theorem can be loosely rephrased as follows: up to compact perturbation, the variational theory of the Hamiltonian action functional of a Hamiltonian on phase space that is quadratic outside a compact set is equivalent to the variational theory of the Riemannian energy functional. This result has numerous dynamical applications and has been reproved in various forms by several authors [1, 3, 2, 54]. Abouzaid presents in Part II of this book yet another proof, which works over $\mathbb{Z}$, does not assume Q to be orientable, and takes into account the BV-algebra structure carried by $\mathcal{L}Q$.

Theorem (Abouzaid). *Let Q be a closed manifold. There is an isomorphism of BV-algebras with integer coefficients*

$$SH^\cdot(T^*Q) \simeq H_\cdot(\mathcal{L}Q; \tau_Q)$$

between the symplectic homology of T^*Q, and the homology of $\mathcal{L}Q$ with coefficients in an explicit local system τ_Q which is trivial if Q is orientable and spin.

Abouzaid's proof is inspired by ideas from family Floer homology and makes use of the canonical Lagrangian fibration structure of T^*Q, a perspective that relates to mirror symmetry. At this point the book definitely crosses the boundary between classical material and new research.

We cannot resist to point out a purely topological view of this result. The cotangent bundle T^*Q with its canonical symplectic form is a symplectic manifold naturally associated to every smooth manifold Q. Therefore one may wonder to what extend the symplectic invariants of this symplectic manifold see the algebraic and differential topology of Q. The theorem clarifies one aspect of this fairly general question, but there is ample room for further research here.

Structure of the book

After this panoramic overview of the context and content of the book, we now present short summaries of each of the chapters, and of the research monograph. This will hopefully help the reader who already knows what she or he is looking for to quickly find his or her way.

Chapter 1 titled *"Basics on free loop spaces"* by David Chataur and Alexandru Oancea, is an introduction to loop spaces, based or free. Its goal is to explain elementary facts about their homotopy theory, topology, and geometry. The authors first discuss homotopical properties (path-loop and loop-loop fibrations, homotopy groups, connected components, homotopy pull-backs). The discussion subsequently focuses on loop spaces as infinite dimensional manifolds, and as an application the authors give a construction of the Chas–Sullivan loop product based on the Thom isomorphism for tubular neighborhoods of Hilbert submanifolds of finite codimension. The diffeology point of view on loop spaces is briefly mentioned. A whole section is dedicated to the Leray–Serre spectral sequence, which is used to perform some explicit homological computations in the case of the path-loop and loop-loop fibrations. The final section discusses orientability of free loop spaces.

Chapter 2 titled *"Morse theory, closed geodesics, and the homology of free loop spaces"* by Alexandru Oancea gives a survey of the existence problem for closed geodesics. The central theme here is the study of the energy functional through variational methods, particularly via Morse theory. The topics that are discussed include: Riemannian background, the Lyusternik–Fet theorem, the Lyusternik–Schnirelmann principle of subordinated classes, the Gromoll–Meyer theorem, Bott's iteration of the index formulas, homological computations using Morse theory, $SO(2)$- vs. $O(2)$-symmetries, Katok's examples and Finsler metrics, and relations to symplectic geometry. The Appendix on *"The problem of existence of infinitely many closed geodesics on the 2-sphere"* by Umberto Hryniewicz gives an account of the proof of the existence of infinitely many closed geodesics on the 2-sphere.

Chapter 3 titled *"Rational homotopy – Sullivan models"* by Luc Menichi is an introduction to Sullivan models from the perspective of rational homotopy theory. Menichi carefully introduces all the algebraic constructions needed to build the Sullivan model of the free loop space of a simply connected manifold. The various building blocks are illustrated in specific examples. These are both of a geometric nature (spheres, H-spaces, projective spaces) and of a conceptual nature (products, fiber products, multiplication, pull-backs). As an application, the author gives the proof of the Vigué-Poirrier–Sullivan theorem, stating that the sequence of rational Betti numbers of the free loop space of a simply connected manifold is unbounded provided the cohomology ring of the manifold is not monogenic.

Chapter 4 titled *"Free loop space and homology"* by Jean-Louis Loday explains the relationship between the (co)homology of the free loop space and the Hochschild homology of its singular cochain algebra. All the relevant technical tools are introduced from scratch, in particular simplicial and cyclic objects, and the chapter sketches the various steps of the proofs, which are otherwise scattered around in the literature. This chapter can be seen as a reading companion to the paper "Cyclic homology and equivariant homology" by J.D.S. Jones [43], which deals with the cyclic case.

Jean-Louis Loday died in tragic circumstances on June 6, 2012. He had given two talks in the 2008–2009 seminar in Strasbourg, enthusiastically embracing the idea that symplectic topologists may get interested in Hochschild and cyclic homology theories. His chapter was submitted in May 2011 and posted on his homepage and on the arXiv in October 2011, indicating that he thought of it as close to final. We have left his work essentially unchanged, except for a few corrections, most of which concern typos or language, and we have addressed the referee's suggestion to include more historical references by inserting several footnotes throughout the text. An appendix, written by Janko Latschev, complements Chapter 4 by a sample computation of Hochschild and cyclic homology groups for the spheres S^r, $r \geq 2$.

Chapter 6 titled *"On algebraic structures of the Hochschild complex"* by Hossein Abbaspour is a study of the algebraic structures carried by the Hochschild (co)homology of a differential graded algebra (dga) under the assumption that it satisfies weak Poincaré duality. Examples of such dga's are Calabi–Yau algebras, derived Poincaré duality algebras and closed Frobenius algebras. The algebraic structures that are discussed include Batalin–Vilkovisky BV-algebra structures on $HH^*(A, A^\vee)$ or $HH^*(A, A)$. The author infers a BV-structure on the homology of the free loop spaces via the theorem of Burghelea–Fiedorowicz–Goodwillie mentioned above. He studies for the first time these BV/coBV structures on Hochschild homology for the case of symmetric open/commutative Frobenius dga's, an inquiry that is motivated by results of Chas–Sullivan [14] and Goresky–Hingston [34] for free loop spaces. The chapter closes with an explanation of the action of Sullivan diagrams on the Hochschild (co)chain complex of a closed Frobenius dga, recovering a result by Tradler–Zeinalian [59] for closed Frobenius algebras.

Chapter 7 titled *"Basic rational string topology"* by Yves Félix gives an introduction to the first string operations, the loop product and the Lie bracket. Much of the material presented in earlier chapters is tied together here. The loop product is presented from three different perspectives: that of the intersection product on a manifold – the original one, that of normal bundles – which connects with the perspective of Chapter 1, and that of shriek maps – which is more algebraic. The author discusses the isomorphism of BV-algebra structures over $\mathbb{Q}$ between the homology of the free loop space and the Hochschild cohomology of the dga of cochains, and this discussion connects to the chapters by Menichi, Loday, and Abbaspour.

Chapter 8 titled *"Fukaya's work on Lagrangian embeddings"* by Janko Latschev discusses some applications of string topology to the study of Lagrangian embeddings into symplectic manifolds, as discovered by Kenji Fukaya [28]. This chapter marks the transition from topology to symplectic geometry, and from strict algebra to homotopical algebra. Fukaya's important observation was that the compactification of the moduli spaces of holomorphic disks with boundary on a Lagrangian submanifold $L \subset \mathbb{C}^n$ can be expressed in terms of string topology operations, specifically the loop bracket (and, depending on the precise implementation, possibly also its higher analogues at the chain level). To derive explicit consequences, the induced Lie algebra up to homotopy on $H.(\mathcal{L}L)$ plays a key role.

The research monograph titled *"Symplectic cohomology and Viterbo's theorem"* by Mohammed Abouzaid discusses a foundational result in symplectic topology, originally due to Viterbo, which connects Floer theory of the cotangent bundle to string topology. The first chapter, numbered as Chapter 9, is a survey which reviews and puts into perspective Floer homology theory. The theory is defined over $\mathbb{Z}$ and the author puts particular emphasis on coherent orientations of moduli spaces and on signs, which play a prominent role in the sequel. The setup chosen by the author is specifically that of the cotangent bundle, but this is only for exposition purposes. The discussion is in fact entirely general and would apply to any Liouville domain. Chapter 10 discusses operations in Floer theory, and proves that Floer homology groups carry the structure of a BV-algebra. Chapter 11 discusses string topology operations from the perspective of finite dimensional approximation, using piecewise geodesics. Finally, the author proves in Chapters 12, 13, and 14 Viterbo's theorem in a new, rather sophisticated version: the symplectic cohomology of the cotangent bundle of a closed manifold M is isomorphic as a BV-algebra over $\mathbb{Z}$ to the homology of its free loop space, the latter being twisted by some specific local system that takes into account the failure of M to be orientable or spin.

Topics that are *not* discussed in this book

There are many topics related to free loop spaces which are barely discussed, if at all, within this book. We list a few of them here since they do bear strong connections with the topics which *are* covered this book, and leave it to the curious reader to discover the relevant literature. The references we mention are just meant as first hints here, and are by no means exhaustive.

- Loops on surfaces [32, 61, 13].
- Moduli spaces of curves [39, 51, 22].
- Operads [30, 49, 47, 27].
- Field theories. String theory [18, 20].
- Loop schemes and (derived) algebraic geometry [44, 6, 7].

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