Introduction

In [31, 32, 33], Floer associated to a non-degenerate time-dependent Hamiltonian

$$H\colon \mathbb{R}/\mathbb{Z} \times M \to \mathbb{R}$$

on a symplectic manifold M (satisfying some technical hypotheses), a cohomology group now called (Hamiltonian) Floer cohomology, which he showed to be independent of H if M is closed.

In these notes, we shall be concerned with a situation where M is not closed. Since general open symplectic manifolds are too wild to allow for an interesting development of Floer theory, one usually restricts attention to those with controlled behaviour outside a compact set; a natural condition to impose is that a neighbourhood of infinity be modelled after the cone on a contact manifold. A key insight of Floer and Hofer [36] is that there are, on such symplectic manifolds, natural classes of Hamiltonians whose Floer cohomology is related to the dynamics of the Reeb flow on the contact manifold at infinity. One such class, which admits a natural order with respect to the "rate of growth" at infinity, was introduced by Viterbo in [81, 82], and the *symplectic cohomology* of such a manifold can be defined as a direct limit of Floer cohomology groups over this class of Hamiltonians. This is the cohomology group appearing in the title. These groups are extremely difficult to compute, except when they vanish, but they are known to satisfy good formal properties, including a version of the Künneth theorem [64].

Instead of considering such a general setting, we restrict ourselves to the first class of examples for which this invariant is both non-trivial and expressible in terms of classical topological invariants: the symplectic manifold M which we shall consider will be the cotangent bundle T^*Q of a closed differentiable manifold. In this case, one naturally obtains a manifold equipped with a contact form by considering the unit sphere bundle with respect to a Riemannian metric on Q, and it has been known for quite a long time that the Reeb flow on this contact manifold is related to the geodesic flow on the tangent bundle. Since the closed orbits of the geodesic flow are the generators of a Morse complex which computes the homology of the free loop space, a connection between the loop homology of Q and the symplectic cohomology of T^*Q is therefore to be expected.

In his ICM address [79], Viterbo explained a strategy for showing that, for cotangent bundles of oriented manifolds, symplectic cohomology is isomorphic to the homology of the free loop space: the idea was to relate both to an intermediate invariant called *generating function homology.* This strategy was implemented in [81], and different approaches were later considered in [3, 71]. Surprisingly, the result stated by Viterbo turns out to be true only if the base is Spin; the key observation here is due to Kragh [50], who showed that, for oriented manifolds, generating function homology cannot be isomorphic to symplectic cohomology because it is not functorial under exact embeddings. Instead, Kragh proved the functoriality of a twisted version of generating function homology, which is isomorphic to the homology of a local system of rank 1 on the free loop space that is trivial if and only the second Stiefel–Whitney class of Q vanishes on all tori. A corrected version of Viterbo's theorem for orientable base was, as a consequence, relatively easy to state and prove [9].

These notes present a complete proof of Viterbo's theorem relating the (twisted) homology of the free loop space of a closed differentiable manifold to the symplectic cohomology of its cotangent bundle. In addition, they include the verification that the primary operadic operations coming on one side from the count of holomorphic curves, and on the other from string topology agree. We pay particular attention to issues of signs and gradings, both because it turns out in the end that the answer is unexpected and because even some experts still consider them to be too mysterious to address.

The original intent was that the account given would be complete as well as accessible to a reader familiar with basic concepts in symplectic topology, but not necessarily an expert. We do not quite succeed in this goal in three respects:

1. The model for the homology of the free loop space that we use is the direct limit of the Morse homology of spaces of piecewise geodesics. This model introduces even more sign conventions that one has to choose and verify are compatible. The choice was made in order to avoid having to reference or prove the fact, well-known to all experts, but with no accessible proof available in the literature, that higher dimensional moduli space of Floer trajectories and their generalisations form manifolds with corners. With such a result at hand, and the additional knowledge that the evaluation map at a fixed point defines a smooth map from such moduli spaces to the ambient symplectic manifold, one would be able to avoid using Morse homology, and rely instead on a more classical theory.
2. While a complete account is given for the construction of a chain map implementing Viterbo's isomorphism, including a verification of the signs in the proof that it is a chain map (see Lemma 3.8 in Chapter 12), the reader who wants to see every detail of the proof that the structure maps coming from Floer theory and string topology are intertwined by this isomorphism will have to do quite a bit of sign checking beyond what is included. Natural orientations are constructed on all moduli spaces that are used to show that the isomorphism preserves operations, but beyond that, one needs to perform some symbol pushing to check that the relations hold as stated, rather than up to an overall sign depending only on discrete invariants (the dimension of Q, the degree of the inputs, …).
3. The construction of a map from Floer theory to loop homology is given in Chapter 12 and one can reasonably hope enough background has been provided that the diligent reader can follow the argument up to that point without being necessarily equipped with expertise in these matters. However, Chapters 13 and 14, in which this map is proved to be an isomorphism, will likely prove to be more challenging because they rely on an essentially new technique using parametrised moduli spaces of pseudoholomorphic curves with Lagrangian boundary conditions.

Beyond the results on the connection between symplectic cohomology and loop homology that have already appeared in the literature (see in particular [82, 5]), several new results are proved. First, statements and proofs are systematically generalised from the orientable to the non-orientable case, including the construction of a natural $\mathbb{Z}$ grading on symplectic cohomology, the definition of string topology operations, and the construction of the isomorphism between (twisted) loop homology and symplectic cohomology.

However, the most important new results are contained in Chapters 13 and 14, which introduce two new mutually inverse maps between loop homology and symplectic cohomology. These maps in a sense explain that Viterbo's theorem holds because

the family of cotangent fibres $\{T_q^*Q\}_{q\in Q}$ defines a Lagrangian foliation of T^*Q.

The motivation for introducing these maps comes from Fukaya's ideas on *family Floer homology*. Moreover, the verification that the maps are mutually inverse uses degenerations of moduli spaces of discs with multiple punctures, which are related to recent work in Floer theory that uses moduli spaces of annuli [41, 17, 8] (see, in particular Figures 13.8 and 14.5). The key point is to

> verify that maps in Floer theory are isomorphisms by considering degenerations of Riemann surfaces, rather than degenerations of Floer equations on a fixed surface.

The idea of degenerating the Floer equation goes back to Floer who used it to prove that certain Floer cohomology groups are isomorphic to ordinary cohomology [33]. Such degenerations usually give rise to *isomorphisms* of chain complexes, but at the cost of requiring very delicate analytic estimates. The method we adopt usually gives a weaker result (only a chain homotopy equivalence), but tends to be more flexible, and requires arguments of a more topological nature.

These notes are organised as follows: symplectic cohomology, with coefficients in a local system over the free loop space, is defined for cotangent bundles in Chapter 9, and three operations on it are constructed in Chapter 10 under the assumption that the local system is *transgressive*. These operations give rise to a (twisted) Batalin–Vilkovisky structure. Chapter 11 is independent of the first two, and provides a construction of a Batalin–Vilkovisky structure on the twisted homology of the loop space of a closed manifold. This structure is constructed from the Morse homology of finite dimensional approximations. A map from symplectic cohomology to loop homology is constructed in Chapter 12, which also includes the verification that this map intertwines the operations on the two sides. A left inverse to this map is constructed in Chapter 13, and Chapter 14 provides the proof that this left inverse is an isomorphism.

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