Introduction

A natural, but slowly emerging program. In his PhD thesis prepared under the supervision of Graham Higman and defended in 1965 [124], and in the article that followed [125], F.A. Garside (1915–1988) solved the Conjugacy Problem for Artin's braid group B_n by introducing a submonoid B_n^+ of B_n and a distinguished element Δ_n of B_n^+ that he called fundamental, and showing that every element of B_n can be expressed as a fraction of the form $\Delta_n^m g$, with m an integer and g an element of B_n^+. Moreover, he proved that any two elements of the monoid B_n^+ admit a least common multiple, thus extending to the non-abelian groups B_n some of the standard tools available in a torsion-free abelian group $\mathbb{Z}^n$.

In the beginning of the 1970's, it was soon realized by E. Brieskorn and K. Saito [36] using an algebraic approach, and by P. Deligne [101] using a more geometric approach, that Garside's results extend to all generalized braid groups associated with finite Coxeter groups, that is, all Artin (or, better, Artin–Tits) groups of spherical type.

The next step forward was the possibility of defining, for every element of the braid monoid B_n^+ (and, more generally, of every spherical Artin–Tits monoid) a distinguished decomposition involving the divisors of the fundamental element Δ_n. The point is that, if g is an element of B_n^+, then there exists a (unique) greatest common divisor g_1 for g and Δ_n and, moreover $g \neq 1$ implies $g_1 \neq 1$. Then g_1 is a distinguished fragment of g (the "head" of g); repeating the operation with g' determined by $g = g_1 g'$, we extract the head g_2 of g' and, iterating, we end up with an expression $g_1 \cdots g_p$ of g in terms of divisors of Δ_n. Although F. Garside was very close to such a decomposition when he proved that greatest common divisors exist in B_n^+, the result does not appear in his work explicitly, and it seems that the first instances of such distinguished decompositions, or *normal forms*, go back to the 1980's in independent work by S. Adyan [2], M. El-Rifai and H. Morton [117], and W. Thurston (circulated notes [226], later appearing as Chapter IX in the book [119] by D. Epstein *et al.*). The normal form was soon used to improve Garside's solution of the Conjugacy Problem [117] and, extended from the monoid to the group, to serve as a paradigmatic example in the then emerging theory of automatic groups of J. Cannon, W. Thurston, and others. Sometimes called the *greedy normal form*—or *Garside normal form*, or *Thurston normal form*—it became a standard tool in the investigation of braids and Artin–Tits monoids and groups from a viewpoint of geometric group theory and of theory of representations, essential in particular in D. Krammer's algebraic proof of the linearity of braid groups [162, 163].

In the beginning of the 1990's, it was realized by one of us that some ideas from F. Garside's approach to braid monoids can be applied in a different context to analyze a certain "geometry monoid" M_{LD} that appears in the study of the so-called left-selfdistributivity law $x(yz) = (xy)(xz)$. In particular, the criterion used by F. Garside to establish that the braid monoid B_n^+ is left-cancellative (that is, $gh = gh'$ implies $h = h'$) can be adapted to M_{LD} and a normal form reminiscent of the greedy normal form exists—with the main difference that the pieces of the normal decompositions are not the divisors of some unique element similar to the Garside braid Δ_n, but they are divisors of elements Δ_T that depend on some object T (actually a tree) attached to the element one wishes to decompose. The approach led to results about the exotic left-selfdistributivity law [73] and, more unexpectedly, about braids and their orderability when it turned out that the monoid M_{LD} naturally projects to the (infinite) braid monoid B_∞^+ [72, 75, 77].

At the end of the 1990's, following a suggestion by L. Paris, the idea arose of listing the abstract properties of the monoid B_n^+ and the fundamental braid Δ_n that make the algebraic theory of B_n possible. This resulted in the notions of a *Garside monoid* and a *Garside element* [99]. In a sense, this is just reverse engineering, and establishing the existence of derived normal decompositions with the expected properties essentially means checking that nothing has been forgotten in the definition. However, it soon appeared that a number of new examples are eligible, and, specially after some cleaning of the definitions was completed [80], that the new framework is really more general than the original braid framework. One benefit of the approach is that extending the results often resulted in discovering new improved arguments no longer relying on superfluous assumptions or specific properties. This program turned out to be rather successful and it led to many developments by a number of different authors [7, 10, 12, 18, 19, 20, 57, 56, 68, 122, 129, 138, 139, 171, 170, 180, 198, 211, ...]. Today the study of Garside monoids is still far from complete, and many questions remain open.

However, in the meanwhile, it soon became clear that, although efficient, the framework of Garside monoids, as stabilized in the 1990s, is far from optimal. Essentially, several assumptions, in particular Noetherianity conditions, are superfluous and they just discard further natural examples. Also, excluding nontrivial invertible elements appears as an artificial limiting assumption. More importantly, one of us (DK) in a 2005 preprint subsequently published as [165] and two of us (FD, JM) [109], as well as David Bessis in an independent research [9], realized that normal forms similar to those involved in Garside monoids can be developed and usefully applied in a context of categories, leading to what they naturally called *Garside categories*. By the way, similar structures are already implicit in the 1976 paper [103] by P. Deligne and G. Lusztig, as well as in the above mentioned monoid M_{LD} [75, 77], and in EG's PhD thesis [134].

It was therefore time around 2007 for the development of a new, unifying framework that would include all the previously defined notions, remove all unneeded assumptions, and allow for optimized arguments. This program was developed in particular during a series of workshops and meetings between 2007 and 2012, and it resulted in the current text. As the above description suggests, the emphasis is put on the normal form and its mechanism, and the framework is that of a general category with only one assumption, namely left-cancellativity. Then the central notion is that of a *Garside family*, defined to be any family that gives rise to a normal form of the expected type. Then, of course, every Garside element Δ in a Garside monoid provides an example of a Garside family, namely the set of all divisors of Δ, but many more Garside families may exist—and they do, as we shall see in the text. Note that, in a sense, our current generalization is the ultimate one since, by definition, no further extension may preserve the existence of a greedy normal form. However, different approaches might be developed, either by relaxing the definition of a greedy decomposition (see the Notes at the end of Chapter III) or, more radically, by placing the emphasis on other aspects of Garside groups rather than on normal forms. Typically, several authors, including J. Crisp, J. McCammond and one of us (DK) proposed to view a Garside group mainly as a group acting on a lattice in which certain intervals of the form $[1, \Delta]$ play a distinguished role, thus paving the way for other types of extensions.

Our hope—and our claim—is that the new framework so constructed is quite satisfactory. By this, we mean that most of the properties previously established in more particular contexts can be extended to larger contexts. It is *not* true that all properties of, say, Garside monoids extend to arbitrary categories equipped with a Garside family but, in most cases, addressing the question in an extended framework helps improving the arguments and really capturing the essential features. Typically, almost all known properties of Garside monoids do extend to categories that admit what we call a bounded Garside family, and the proofs cover for free all previously considered notions of Garside categories.

It is clear that a number of future developments will continue to involve particular types of monoids or categories only: we do not claim that our approach is universal... However, we would be happy if the new framework—and the associated terminology—could become a useful reference for further works.

About this text. The aim of the current text is to give a state-of-the-art presentation of this approach. Finding a proper name turned out to be not so obvious. On the one hand, “Garside calculus” would be a natural title, as the greedy normal form and its variations are central in this text: although algorithmic questions are not emphasized, most constructions are effective and the mechanism of the normal form is indeed a sort of calculus. On the other hand, many results, in particular those of structural nature, exploit the normal form but are not reducible to it, making a title like “Garside structures” or “Garside theory” more appropriate. But such a title is certainly too ambitious for what we can offer: no genuine structure theory or no exhaustive classification of, say, Garside families, is to be expected at the moment. What we do here is develop a framework that, we think and hope, can become a good base for a still-to-come theory. Another option could have been “Garside categories”, but it will be soon observed that no notion with that name is introduced here: in view of the subsequent developments, a reasonable meaning could be “a cancellative category that admits a Garside map”, but a number of variations are still possible, and any particular choice could become quickly obsolete—as is, in some sense, the notion of a Garside group. Finally, we hope that our current title, “Foundations of Garside Theory”, reflects the current content in a proper way: the current text is an invitation to further research, and does not aim at being exhaustive—reporting about all previous results involving Garside structures would already be very difficult—but concentrates on what seems to be the core of the subject.

The text in divided into two parts. Part A is devoted to general results and offers a careful treatment of the bases. Here complete proofs are given, and the results are illustrated with a few basic examples. By contrast, Part B consists of essentially independent chapters explaining further examples, or families of examples, that are in general more elaborate. Here some proofs are omitted, and the discussion is centered around what can be called the Garside aspects in the considered structures.

Our general scheme will be to start from an analysis of normal decompositions and then introduce Garside families as the framework guaranteeing the existence of normal decompositions. Then the three main questions we shall address and a chart of the corresponding chapters looks as follows:

- **How** do Garside structures work? (mechanism of normal decomposition)
    - Chapter III (domino rules, geometric aspects)
    - Chapter VII (compatibility with subcategories)
    - Chapter VIII (connection with conjugacy)
- **When** do Garside structures exist? (existence of normal decomposition)
    - Chapter IV (recognizing Garside families)
    - Chapter VI (recognizing Garside germs)
    - Chapter V (recognizing Garside maps)
- **Why** consider Garside structures? (examples and applications)
    - Chapter I (basic examples)
    - Chapter IX (braid groups)
    - Chapter X (Deligne–Lusztig varieties)
    - Chapter XI (selfdistributivity)
    - Chapter XII (ordered groups)
    - Chapter XIII (Yang–Baxter equation)
    - Chapter XIV (four more examples)

Above, and in various places, we use "Garside structure" as a generic and informal way to refer to the various objects occurring with the name "Garside": Garside families, Garside groups, Garside maps, *etc*.

The chapters. To make further reference easy, each chapter in Part A begins with a summary of the main results. At the end of each chapter, exercises are proposed, and a Notes section provides historical references, comments, and questions for further research.

Chapter I is introductory and lists a few examples. The chapter starts with classical examples of Garside monoids, such as free abelian monoids or classical and dual braid monoids, and it continues with some examples of structures that are not Garside monoids, but nevertheless possess a normal form similar to that of Garside monoids, thus providing a motivation for the construction of a new, extended framework.

Chapter II is another introductory chapter in which we fix some terminology and basic results about categories and derived notions, in particular connected with divisibility relations that play an important rôle in the sequel. A few general results about Noetherian categories and groupoids of fractions are established. The final section describes an general method, called reversing, for investigating a presented category. As the question is not central in our current approach (and although it owes much to F.A. Garside's methods), some proofs of this section are deferred to the Appendix at the end of the book.

Chapter III is the one where the theory really starts. Here the notion of a normal decomposition is introduced, as well as the notion of a Garside family, abstractly introduced as a family that guarantees the existence of an associated normal form. The mechanism of the normal form is analyzed, both in the case of a category ("positive case") and in the case of its enveloping groupoid ("signed case"): some simple diagrammatic patterns, the domino rules, are crucial, and their local character directly implies various geometric consequences, in particular a form of automaticity and the Grid Property, a strong convexity statement.

Chapter IV is devoted to obtaining concrete characterizations of Garside families, in other words, conditions that guarantee the existence of normal decompositions. In this chapter, one establishes external characterizations, meaning that we start with a category $\mathcal{C}$ and look for conditions ensuring that a given subfamily $\mathcal{S}$ of $\mathcal{C}$ is a Garside family. Various answers are given, in a general context first, and then in particular contexts where some conditions come for free: typically, if the ambient category $\mathcal{C}$ is Noetherian and admits unique least common right-multiples, then a subfamily $\mathcal{S}$ of $\mathcal{C}$ is a Garside family if and only if it generates $\mathcal{C}$ and is closed under least common right-multiple and right-divisor.

In Chapter V, we investigate particular Garside families that are called bounded. Essentially, a Garside family $\mathcal{S}$ is bounded is there exists a map Δ (an element in the case of a monoid) such that $\mathcal{S}$ consists of the divisors of Δ (in some convenient sense). Not all Garside families are bounded, and, contrary to the existence of a Garside family, the existence of a bounded Garside family is not guaranteed in every category. Here we show that a bounded Garside family is sufficient to prove most of the results previously established for a Garside monoid, including the construction of Δ-normal decompositions, a variant of the symmetric normal decompositions used in groupoids of fractions.

Chapter VI provides what can be called internal (or intrinsic) characterizations of Garside families: here we start with a family $\mathcal{S}$ equipped with a partial product, and we wonder whether there exists a category $\mathcal{C}$ in which $\mathcal{S}$ embeds as a Garside family. The good news is that such characterizations do exist, meaning that, when the conditions are satisfied, all properties of the generated category can be read inside the initial family $\mathcal{S}$. This local approach turns to be useful for constructing examples and, in particular, it can be used to construct a sort of unfolded, torsion-free version of convenient groups, typically braid groups starting from Coxeter groups.

Chapter VII is devoted to subcategories. Here one investigates natural questions such as the following: if $\mathcal{S}$ is a Garside family in a category $\mathcal{C}$ and $\mathcal{C}_1$ is a subcategory of $\mathcal{C}$, then is $\mathcal{S} \cap \mathcal{C}_1$ a Garside family in $\mathcal{C}_1$ and, if so, what is the connection between the associated normal decompositions? Of particular interest are the results involving subgerms, which provide a possibility of reading inside a given Garside family $\mathcal{S}$ the potential properties of the subcategories generated by the subfamilies of $\mathcal{S}$.

In Chapter VIII, we address conjugacy, first in the case of a category equipped with an arbitrary Garside family, and then, mainly, in the case of a category equipped with a bounded Garside family. Here again, most of the results previously established for Garside monoids can be extended, including the cycling, decycling, and sliding transformations which provide a decidability result for the Conjugacy Problem whenever convenient finiteness assumptions are satisfied. We also extend the geometric methods of Bestvina to describe periodic elements in this context.

Part B begins with Chapter IX, devoted to (generalized) braid groups. Here we show how both the reversing approach of Chapter II and the germ approach of Chapter VI can be applied to construct and analyze the classical and dual Artin–Tits monoids. We also mention the braid groups associated with complex reflection groups, as well as several exotic Garside structures on B_n. The applications of Garside structures in the context of braid groups are too many to be described exhaustively, and we just list some of them in the Notes section.

Chapter X is a direct continuation of Chapter IX. It reports about the use of Garside-type methods in the study of Deligne–Lusztig varieties, an ongoing program that aims at establishing by a direct proof some of the consequences of the Broué Conjectures about finite reductive groups. Several questions in this approach directly involve conjugacy in generalized braid groups, and the results of Chapter VIII are then crucial.

Chapter XI is an introduction to the Garside structure hidden in the above mentioned algebraic law $x(yz) = (xy)(xz)$, a typical example where a categorical framework is needed (or, at the least, the framework of Garside monoids is not sufficient). Here a promising contribution of the Garside approach is a natural program possibly leading to the so-called Embedding Conjecture, a deep structural result so far resisting all attempts.

In Chapter XII, we develop an approach to ordered groups based on divisibility properties and Garside elements, resulting in the construction of groups with the property that the associated space of orderings contains isolated points, which answers one of the natural questions of the area. Braid groups are typical examples, but considering what we call triangular presentations leads to a number of different examples.

Chapter XIII is a self-contained introduction to set-theoretic solutions of the Yang–Baxter equation and the associated structure groups, an important family of Garside groups. The exposition is centered on the connection between the RC-law $(xy)(xz) = (yx)(yz)$ and the right-complement operation on the one hand, and what is called the geometric I-structure on the other hand. Here the Garside approach both provides a specially efficient framework and leads to new results.

In Chapter XIV, we present four unrelated topics involving interesting Garside families: divided categories and decompositions categories with two applications, then an extension of the framework of Chapter XIII to more general RC-systems, then what is called the braid group of $\mathbb{Z}^n$, a sort of analog of Artin's braid group in which permutations of $\{1, ..., n\}$ are replaced with linear orderings of $\mathbb{Z}^n$, and, finally, an introduction to groupoids of cell decompositions that arise when the mapping class group approach to braid groups is extended by introducing sort of roots of the generators σ_i.

The final Appendix contains the postponed proofs of some technical statements from Chapter II for which no complete reference exists in literature.

Exercises are proposed at the end of most chapters. Solutions to some of them, as well as a few proofs from the main text that are skipped in the book, can be found at the address

`www.math.unicaen.fr/~garside/Addenda.pdf`

as well as in arXiv:1412.5299.

Thanks. We primarily wish to thank David Bessis, who was part of the crew at an early stage of the project, but then quit it for personal reasons.

Next, we thank all the colleagues and students who participated in the various meetings dedicated to the project and who contributed by their questions and suggestions to this text. A (certainly non-exhaustive) list includes Marc Autord, Serge Bouc, Michel Broué, Matthieu Calvez, Ruth Corran, Pierre-Louis Curien, Matthew Dyer, Jean Fromentin, Volker Gebhardt, Tomas Gobet, Juan González-Meneses, Yves Guiraud, Tatiana Ivanova–Gateva, Jean-Yves Hée, Christophe Hohlweg, Eric Jespers, Eon-Kyung Lee, Sang-Jin Lee, Gilbert Levitt, Jon McCammond, Ivan Marin, Paul-André Melliès, François Métayer, Jan Okniński, Luis Paris, Matthieu Picantin, Wolfgang Rump, Elena Sendroiu, Maya Van Campenhout, Bert Wiest. Also, special thoughts go to Joan Birman and Hugh Morton, whose support and interest in the subject has always been strong.

The project was supported partly by the French ANR grant TheoGar ANR-08-BLAN-0269-02.

Patrick Dehornoy
François Digne
Eddy Godelle
Daan Krammer
Jean Michel

About notation

We use

- $\mathbb{N}$: set of all nonnegative integers
- $\mathbb{Z}$: set of all integers
- $\mathbb{Q}$: set of all rational numbers
- $\mathbb{R}$: set of all real numbers
- $\mathbb{C}$: set of all complex numbers

As much as possible, different letters are used for different types of objects, according to the following list:

- $\mathcal{C}$: category
- $\mathcal{S}, \mathcal{X}$: generic subfamily of a category
- $\mathcal{A}$: atom family in a category
- x, y, z: generic object of a category
- f, g, h: generic element (morphism) in a category, a monoid, or a group
- a, b, c, d, e: special element in a category (atom, endomorphism, etc.)
- ϵ: invertible element in a category
- r, s, t: element of a distinguished subfamily (generating, Garside, ...)
- i, j, k : integer variable (indices of sequences)
- ℓ, m, n, p, q: integer parameter (fixed, for instance, length of a sequence)
- $\mathtt{a}, \mathtt{b}, \mathtt{c}$: constant element of a category or a monoid for concrete examples (a special notation to distinguish from variables)
- u, v, w : path (or word)
- α, β, γ : binary address (in terms and binary trees)
- ϕ, ψ, π : functor