

# Contents

<b>A General theory</b>	<b>1</b>
<b>I Some examples</b>	<b>3</b>
1 Classical examples . . . . .	3
1.1 Free abelian monoids . . . . .	3
1.2 Braid groups and monoids . . . . .	5
1.3 Dual braid monoids . . . . .	10
2 Garside monoids and groups . . . . .	12
2.1 The notion of a Garside monoid . . . . .	12
2.2 More examples . . . . .	14
3 Why a further extension? . . . . .	15
3.1 Infinite braids . . . . .	15
3.2 The Klein bottle group . . . . .	17
3.3 Wreathed free abelian groups . . . . .	19
3.4 Ribbon categories . . . . .	20
<b>II Preliminaries</b>	<b>27</b>
1 The category context . . . . .	29
1.1 Categories and monoids . . . . .	29
1.2 Subfamilies and subcategories . . . . .	31
1.3 Invertible elements . . . . .	32
1.4 Presentations . . . . .	37
2 Divisibility and Noetherianity . . . . .	40
2.1 Divisibility relations . . . . .	40
2.2 Lcms and gcds . . . . .	42
2.3 Noetherianity conditions . . . . .	45
2.4 Height . . . . .	51
2.5 Atoms . . . . .	55
3 Groupoids of fractions . . . . .	57
3.1 The enveloping groupoid of a category . . . . .	58
3.2 Groupoid of fractions . . . . .	60
3.3 Torsion elements in a groupoid of fractions . . . . .	63
4 Working with presented categories . . . . .	64
4.1 A toolbox . . . . .	64
4.2 Right-reversing: definition . . . . .	69
4.3 Right-reversing: termination . . . . .	72
4.4 Right-reversing: completeness . . . . .	76

<b>III</b>	<b>Normal decompositions</b>	91
1	Greedy decompositions . . . . .	94
1.1	The notion of an $\mathcal{S}$ -greedy path . . . . .	94
1.2	The notion of an $\mathcal{S}$ -normal path . . . . .	98
1.3	The notion of a Garside family . . . . .	103
1.4	Recognizing Garside families . . . . .	105
1.5	The second domino rule . . . . .	113
2	Symmetric normal decompositions . . . . .	117
2.1	Left-disjoint elements . . . . .	118
2.2	Symmetric normal decompositions . . . . .	121
2.3	Uniqueness of symmetric normal decompositions . . . . .	123
2.4	Existence of symmetric normal decompositions . . . . .	125
2.5	Computation of symmetric normal decompositions . . . . .	128
3	Geometric and algorithmic properties . . . . .	139
3.1	Geodesics . . . . .	139
3.2	The Grid Property . . . . .	140
3.3	The Fellow Traveller Property . . . . .	145
3.4	The Garside resolution . . . . .	150
3.5	Word Problem . . . . .	160
<b>IV</b>	<b>Garside families</b>	169
1	The general case . . . . .	172
1.1	Closure properties . . . . .	172
1.2	Characterizations of Garside families . . . . .	179
1.3	Special Garside families . . . . .	182
1.4	Head functions . . . . .	186
2	Special contexts . . . . .	190
2.1	Solid families . . . . .	190
2.2	Right-Noetherian categories . . . . .	193
2.3	Categories that admit right-mcms . . . . .	198
2.4	Categories with unique right-lcms . . . . .	204
2.5	Finite height . . . . .	206
3	Geometric and algorithmic applications . . . . .	208
3.1	Presentations . . . . .	208
3.2	Word Problem . . . . .	212
3.3	The case of categories with lcms . . . . .	215
<b>V</b>	<b>Bounded Garside families</b>	227
1	Right-bounded Garside families . . . . .	229
1.1	The notion of a right-bounded Garside family . . . . .	230
1.2	Right-Garside maps . . . . .	233
1.3	The functor $\phi_\Delta$ . . . . .	236
1.4	Powers of a right-bounded Garside family . . . . .	239
1.5	Preservation of normality . . . . .	242
2	Bounded Garside families . . . . .	245
2.1	The notion of a bounded Garside family . . . . .	245

2.2	Powers of a bounded Garside family . . . . .	248
2.3	The case of a cancellative category . . . . .	250
2.4	Garside maps . . . . .	252
2.5	Existence of lcms and gcds . . . . .	254
3	Delta-normal decompositions . . . . .	257
3.1	The positive case . . . . .	258
3.2	The general case . . . . .	261
3.3	Symmetric normal decompositions . . . . .	268
3.4	Co-normal decompositions . . . . .	270
<b>VI</b>	<b>Germs</b>	<b>277</b>
1	Germs . . . . .	280
1.1	The notion of a germ . . . . .	280
1.2	The embedding problem . . . . .	282
1.3	Garside germs . . . . .	287
2	Recognizing Garside germs . . . . .	290
2.1	The families $\mathcal{I}$ and $\mathcal{J}$ . . . . .	291
2.2	Greatest $\mathcal{I}$ -functions . . . . .	298
2.3	Noetherian germs . . . . .	300
2.4	An application: germs derived from a groupoid . . . . .	304
3	Bounded germs . . . . .	310
3.1	Right-bounded germs . . . . .	310
3.2	Bounded germs . . . . .	312
3.3	An application: germs from lattices . . . . .	315
<b>VII</b>	<b>Subcategories</b>	<b>319</b>
1	Subcategories . . . . .	321
1.1	Closure properties . . . . .	322
1.2	Subcategories that are closed under $=^x$ . . . . .	326
1.3	Head subcategories . . . . .	327
1.4	Parabolic subcategories . . . . .	332
2	Compatibility with a Garside family . . . . .	335
2.1	Greedy paths . . . . .	335
2.2	Compatibility with a Garside family . . . . .	337
2.3	Compatibility, special subcategories . . . . .	340
2.4	Compatibility with symmetric decompositions . . . . .	342
3	Subfamilies of a Garside family . . . . .	345
3.1	Subgerms . . . . .	345
3.2	Transitivity of closure . . . . .	348
3.3	Garside subgerms . . . . .	351
4	Subcategories associated with functors . . . . .	357
4.1	Subcategories of fixed points . . . . .	357
4.2	Image subcategories . . . . .	359

<b>VIII Conjugacy</b>	<b>371</b>
1 Conjugacy categories . . . . .	373
1.1 General conjugacy . . . . .	373
1.2 Cyclic conjugacy . . . . .	378
1.3 Twisted conjugacy . . . . .	383
1.4 An example: ribbon categories . . . . .	387
2 Cycling, sliding, summit sets . . . . .	395
2.1 Cycling and decycling . . . . .	396
2.2 Sliding circuits . . . . .	404
3 Conjugacy classes of periodic elements . . . . .	416
3.1 Periodic elements . . . . .	416
3.2 Geometric methods . . . . .	419
3.3 Conjugacy of periodic elements . . . . .	426
<b>B Specific examples</b>	<b>435</b>
<b>IX Braids</b>	<b>437</b>
1 The classical Garside structure on Artin–Tits groups . . . . .	437
1.1 Coxeter groups . . . . .	438
1.2 Artin–Tits groups, reversing approach . . . . .	443
1.3 The germ approach . . . . .	449
2 More Garside structures on Artin–Tits groups . . . . .	452
2.1 The dual braid monoid . . . . .	452
2.2 The case of the symmetric group . . . . .	454
2.3 The case of finite Coxeter groups . . . . .	458
2.4 Exotic Garside structures on $B_n$ . . . . .	459
3 Braid groups of well-generated complex reflection groups . . . . .	462
3.1 Complex reflection groups . . . . .	462
3.2 Braid groups of complex reflection groups . . . . .	464
3.3 Well-generated complex reflection groups . . . . .	465
3.4 Tunnels . . . . .	466
3.5 The Lyashko–Looijenga covering and Hurwitz action . . . . .	468
<b>X Deligne–Lusztig varieties</b>	<b>477</b>
1 Finite reductive groups . . . . .	477
1.1 Reductive groups . . . . .	478
1.2 Some important subgroups . . . . .	478
1.3 $G^F$ -conjugacy . . . . .	479
2 Representations . . . . .	481
2.1 Complex representations of $G^F$ . . . . .	481
2.2 Deligne–Lusztig varieties . . . . .	481
2.3 Modular representation theory . . . . .	483
3 Geometric Broué Conjecture, torus case . . . . .	485
3.1 The geometric approach . . . . .	485
3.2 Endomorphisms of Deligne–Lusztig varieties . . . . .	486

3.3	Periodic elements . . . . .	488
4	Geometric Broué Conjecture, the general case . . . . .	490
4.1	The parabolic case . . . . .	490
4.2	The really general case . . . . .	493
<b>XI</b>	<b>Left self-distributivity</b>	499
1	Garside sequences . . . . .	500
1.1	Partial actions . . . . .	500
1.2	Right-Garside sequences . . . . .	502
1.3	Derived notions . . . . .	505
2	LD-expansions and the category $\mathcal{LD}_0$ . . . . .	507
2.1	Free LD-systems . . . . .	507
2.2	LD-expansions . . . . .	509
2.3	The category $\mathcal{LD}_0$ . . . . .	511
2.4	Simple LD-expansions . . . . .	512
3	Labeled LD-expansions and the category $\mathcal{LD}$ . . . . .	513
3.1	The operators $\Sigma_\alpha$ . . . . .	513
3.2	The monoid $M_{\text{LD}}$ . . . . .	515
3.3	The category $\mathcal{LD}$ . . . . .	518
3.4	The Embedding Conjecture . . . . .	521
4	Connection with braids . . . . .	524
4.1	The main projection . . . . .	525
4.2	Reproving braid properties . . . . .	527
4.3	Hurwitz action of braids on LD-systems . . . . .	530
<b>XII</b>	<b>Ordered groups</b>	537
1	Ordered groups and monoids of $O$ -type . . . . .	537
1.1	Orderable and bi-orderable groups . . . . .	538
1.2	The spaces of orderings on a group . . . . .	541
1.3	Two examples . . . . .	545
2	Construction of isolated orderings . . . . .	546
2.1	Triangular presentations . . . . .	546
2.2	Existence of common multiples . . . . .	549
2.3	More examples . . . . .	553
2.4	Effectivity questions . . . . .	554
3	Further results . . . . .	557
3.1	Dominating elements . . . . .	557
3.2	Right-ceiling . . . . .	558
3.3	The specific case of braids . . . . .	560
<b>XIII</b>	<b>Set-theoretic solutions of Yang–Baxter equation</b>	567
1	Several equivalent frameworks . . . . .	567
1.1	Set-theoretic solutions of YBE . . . . .	568
1.2	Involutive biracks . . . . .	570
1.3	RC- and RLC-quasigroups . . . . .	572
2	Structure monoids and groups . . . . .	577

2.1	Structure monoids and groups . . . . .	578
2.2	RC-calculus . . . . .	581
2.3	Every structure monoid is a Garside monoid . . . . .	586
2.4	A converse connection . . . . .	589
3	<i>I</i> -structure . . . . .	591
3.1	From RC-quasigroups to <i>I</i> -structures . . . . .	591
3.2	From <i>I</i> -structures to RC-quasigroups . . . . .	594
3.3	Coxeter-like groups . . . . .	599
<b>XIV</b>	<b>More examples</b>	611
1	Divided and decomposition categories . . . . .	611
1.1	Divided categories . . . . .	612
1.2	Decomposition categories . . . . .	617
2	Cyclic systems . . . . .	624
2.1	Weak RC-systems . . . . .	625
2.2	Units and ideals . . . . .	627
2.3	The structure category of a weak RC-system . . . . .	631
3	The braid group of $\mathbb{Z}^n$ . . . . .	636
3.1	Ordering orders . . . . .	637
3.2	Lexicographic orders of $\mathbb{Z}^n$ . . . . .	638
3.3	A lattice ordering on $GL(n, \mathbb{Z})$ . . . . .	639
4	Cell decompositions of a punctured disk . . . . .	641
4.1	Braid groups as mapping class groups . . . . .	641
4.2	Cell decompositions . . . . .	643
4.3	The group $B_\ell$ and the category $\mathcal{B}_\ell$ . . . . .	644
4.4	Flips . . . . .	647
4.5	A bounded Garside family . . . . .	651
<b>Appendix</b>		657
1	Groupoids of fractions . . . . .	657
2	Working with presented categories . . . . .	663
<b>Bibliography</b>		671
<b>Index</b>		685